Quantitative methods for business decisions - exercise classes

Prudky, Ivan

QUANTITATIVE METHODS FOR BUSINESS DECISIONS – EXERCISE CLASSES 2020/2021

Ivan Prudky
ivan.prudky@gmail.com

Mathematical formulation of linear programming problem

Exercises 1.

International Business

ILO1	convert the practical problem of optimization into a mathematical statement as well as interpret results of applied quantitative models in business decision making

Linear programming: Why do we use it?

First steps…

- **Field of MATHEMATICS** → deals with the problem of system optimization within given / set limits

- *Leonid Kantorovich* → 1975 together with *Koopmans:*

Nobel prize in economics

First steps…

- **USA:**

for military logistics problems

→ optimizing army and military equipment transportation

TODAY:

→most commonly used quantitative **optimization method**

LP usage

→ different problems, but three basic **elements**:

1. a set of **decisions** to be taken;
2. the **goal** to be maximized or minimized - depending on the nature of the problem being solved;
3. a set of **constraints** that introduce certain restrictions when deciding.

LP usage

USAGE	DECISION	OBJECTIVE (GOAL)	LIMITATIONS
Planning of production	How to produce a product?	Maximum Total Revenue	▪ materials ▪ equipment ▪ labour force
Investment planning	How to invest leftover capital?	Maximize your annual fund yield	▪ amount of money ▪ legal frames ▪ investment risks
Distribution (transportation) of goods	How to distribute products (by type, quantity)?	Minimize transport costs	▪ quantity of goods ▪ means of transport ▪ demand
Advertising planning	How to advertise in the media (by type, by quantity)?	Minimize costs or maximize advertising performance	▪ amount of money ▪ time ▪ available media
Labour force placement planning	How to allocate working hours / individual jobs?	Minimize workforce costs	▪ quantity of working hours ▪ number of employees ▪ legal frame (worker union)

Mathematical formulation of linear programming problem

- → Objective function, constraints, non-negativity condition
- → General and standard form of the model
- → Structural and slack variables

Mathematical formulation of linear programming problem

→ Objective function, constraints, non-negativity condition

OF: → the criteria for evaluating the solution to the problem considered

C: →a series of functions (mathematical equations / inequalities) that describe physical, economic, technological, legal or ethical restrictions that have decision variables

NN: →decision-making variables can not be negative (* we produce -50 products)

Mathematical formulation of linear programming problem

→ Canonical model form

→ converting of linear inequalities into equations by adding additional / slack / equalizing variables

→ Structural and slack variables

→ Structural variables = the variables we already have in the model

→ Slack variables = surpluses / deficits

EXERCISE 1

EXERCISE 1.

- A person invests **300.000,00 HRK** in two funds, **F1** and **F2**. The investor requires from a broker to invest **a maximum of 120.000,00 HRK** in fund **F2** and **at least 60.000,00 HRK** in fund **F1**. He also wants the amount invested in fund **F1 to be bigger or at least equal** to the amount invested in fund **F2**. The expected profit of the **F1** fund is **8 %** and the fund **F2 12 %**.
- What should the broker advise the investor (how much money should be invested in fund **F1** and how much in fund **F2**) to **make the most of the profit**?

a) Mathematically formulate this problem of linear programming.

b) Write the standard form of this problem and explain the meaning of structural and slack variables.

EXERCISE 1.

	Fund F1	Fund F2	Constraints
Available money	1	1	300.000,00 kn
Max investment F2	0	1	120.000,00 kn
Min investment F1	1	0	60.000,00 kn
Investors requirement	1	1	0
Expected profit (%)	8	12	

EXERCISE 1.

a) Mathematically formulate this problem of linear programming.

GENERAL FORM:

* $Max \rightarrow \leq$

$Min \rightarrow \geq$

$$MaxZ = 0{,}08x_1 + 0{,}12x_2$$

objective function

$$\left.\begin{aligned} x_1 + x_2 &\leq 300.000 \\ x_2 &\leq 120.000 \\ x_1 &\geq 60.000 \\ x_1 - x_2 &\geq 0 \end{aligned}\right\} \text{constraints}$$

$$x_1, x_2 \geq 0$$

non-negativity

EXERCISE 1.

b) Write the standard form of this problem and explain the meaning of structural and slack variables.

CANONICAL FORM:

$$MaxZ = 0{,}08x_1 + 0{,}12x_2 \boxed{+ 0x_3 + 0x_4 + 0x_5 + 0x_6}$$

$$\begin{aligned} x_1 + x_2 + x_3 &= 300.000 \\ x_2 + x_4 &= 120.000 \\ x_1 - x_5 &\geq 60.000 \\ x_1 - x_2 + x_6 &\geq 0 \end{aligned}$$

≥
slack variables
are subtrackted

≤
slack variables
are added up

$$x_1, x_2, \boxed{x_3, x_4, x_5, x_6} \geq 0$$

EXERCISE 1.

b) Write the standard form of this problem and explain the meaning of structural and slack variables.

$x_1, x_2 \rightarrow structura\ variables$

$x_3, x_4, x_5, x_6 \rightarrow slack\ variables$

$\leq \rightarrow UNUSED$

$\geq \rightarrow OVERFLOW$

x_1: amount of money invested into fund F1 (kn)

x_2: amount of money invested into fund F2 (kn)

x_3: unused amount of money from the available budget (kn)

x_4: unused amount of money which could have been invested into fund F2 (kn)

x_5: amount of money above the minimum required investment into fund F1 (kn)

x_6: exceeded value of investment into fund F1 above the investors requirement (kn)

EXERCISE 2

EXERCISE 2.

- The company COOL sets up two electrical products: **air conditioning units** and **special fans** for a known customer.
- For the assembling of one of the air conditioning units, as well as for the assembling of one special fan, it takes **15 minutes**, and the company has a day with 250 working hours for product assembling tasks.
- The time for quality control and packaging of the air conditioner is **9 minutes**, and for the special fan unit **18 minutes**, wherein the daily operating hours available for quality control and packaging are **210**.
- Each special fan is fitted with one propeller, and the company's warehouse can provide 600 propellers per day. The customer asks that **at least 20 % of all delivered products are special fans**.
- If the company's profit is 15 EUR for the delivered air conditioner and 20 EUR for the special fan delivered, specify the daily production schedule of the air conditioning units or special fans that will give COOL **the highest profit**.

EXERCISE 2.

	Air conditioning unit	Special fan	Constraints
Assembling time	15 min	15 min	250 h
Quality control	9 min	18 min	210 h
Q of propellers		1	600 kom
Min. Q of special fans		1	20 % of all
Profit	15 €	20 €	

EXERCISE 2.

a) Mathematically formulate this problem of linear programming.

GENERAL MODEL:

$$\boldsymbol{Max = 15x_1 + 20x_2}$$

$$15x_1 + 15x_2 \leq 15.000$$

$$9x_1 + 18x_2 \leq 12.600$$

$$x_2 \leq 600$$

$$x_2 \geq 0{,}2(x_1 + x_2) \quad \rightarrow \quad 0{,}8x_2 - 0{,}2x_1 \geq 0$$

$$x_1, x_2 \geq 0$$

EXERCISE 2.

b) Write the standard form of this problem and explain the meaning of structural and slack variables.

CANONICAL MODEL:

$$\boldsymbol{Max = 15x_1 + 20x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6}$$

$$15x_1 + 15x_2 + x_3 = 15.000$$

$$9x_1 + 18x_2 + x_4 = 12.600$$

$$x_2 + x_5 = 600$$

$$0{,}8x_2 - 0{,}2x_1 - x_6 = 0$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

EXERCISE 2.

b) Write the standard form of this problem and explain the meaning of structural and slack variables.

x_1: number of air conditioning units sold

x_2: number of special fans sold

x_3: unused working minutes for assembling (min)

x_4: unused working minutes for quality control and packaging (min)

x_5: unused amount of propelers on stock

x_6: the amount of special fans exeeding the customer requirement

EXERCISE 3

EXERCISE 3.

- The company manufactures two types of products, **P1** and **P2**, on two different machines, **S1** and **S2**.
- For **P1** production, 1 hour of machine **S1** and 0.5 hours of machine **S2** work is required, while **P2** requires 1 hour of machine **S1** and 1.5 hours of machine **S2** work. The available daily capacity of the **S1** machine is 16 hours, and the **S2** machine is 12 hours.
- In one **P1** product unit are 2 kilograms of **M1** material and 1 kg of **M2** material incorporated, while 1 kilogram of **M1** material is incorporated into product **P2**. 20 kg of **M1** material and 8 kg of **M2** material are on stock at the warehouse.
- The profit per product **P1** amounts to 120.00 HRK, and per product **P2** 80.00 HRK, whereby the buyer requests from the manufacturer that the quantity of product **P1** is at least 20% of the quantity of product **P2**. Determine the daily production schedule of P1 and P2 products that will **maximize company profits**.

a) Mathematically formulate this problem of linear programming.

b) Write the standard form of this problem and explain the meaning of structural and slack variables.

EXERCISE 3.

	Pro1	Pro2	Constraint
Machine S001	1 h	1 h	16 h
Machine S002	0,5 h	1,5 h	12 h
Material MI1	2 kg	1 kg	20 kg
Material MI2	1 kg		8 kg
Customer requir.	1		≥ 20 % Pro 2
Profit	120,00 kn	80,00 kn	

EXERCISE 3.

a) Mathematically formulate this problem of linear programming.

GENERAL MODEL:

$$\boldsymbol{MaxZ = 120x_1 + 80x_2}$$

$$x_1 + x_2 \leq 16$$
$$0{,}5x_1 + 1{,}5x_2 \leq 12$$
$$2x_1 + 1x_2 \leq 20$$
$$x_1 \leq 8$$
$$x_1 \geq 0{,}2x_2$$
$$x_1, x_2 \geq 0$$

EXERCISE 3.

b) Write the standard form of this problem and explain the meaning of structural and slack variables.

CANONICAL MODEL:

$$\boldsymbol{MaxZ = 120x_1 + 80x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6 + 0x_7}$$

$$x_1 + x_2 + x_3 = 16$$
$$0{,}5x_1 + 1{,}5x_2 + x_4 = 12$$
$$2x_1 + 1x_2 + x_5 = 20$$
$$x_1 + x_6 = 8$$
$$x_1 - x_7 = 0{,}2x_2$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$$

EXERCISE 3.

b) Write the standard form of this problem and explain the meaning of structural and slack variables.

x_1: Pro1 product quantity

x_2: Pro2 product quantity

x_3: unused machine hours S001 (h)

x_4: unused machine hours S002 (h)

x_5: unused amount of MI1 material (kg)

x_6: unused amount of MI2 material (kg)

x_7: the amount of product Pro1 delivered exeeding the buyers requirement

EXERCISE 4

EXERCISE 4.

- The company produces four products: **PA, PB, PC** and **PD**. The final parts of the process of making the products are assembling, polishing and packaging operations. The time required to perform each of the above operations in minutes is shown in the table below. The same table shows the profit per piece of each product.

Product	Assembling	Polishing	Packaging	Profit (EUR)
PA	2	3	2	1,50
PB	4	2	3	2,50
PC	3	3	2	3,00
PD	7	4	5	4,50

EXERCISE 4.

- The company annually has 100.000 minutes for the assembling process, 50.000 minutes for polishing and 60.000 minutes for packaging.
- Determine the annual production plan of certain products for which the company will **earn the highest profit**.

a) Mathematically formulate this problem of linear programming.

b) Write the standard form of this problem and explain the meaning of structural and slack variables.

EXERCISE 4.

	PA	PB	PC	PD	Constraint
Assembling (min)	2	4	3	7	100.000 min
Polishing (min)	3	2	3	4	50.000 min
Packaging (min)	2	3	2	5	60.000 min
Profit (€)	1,50	2,50	3,00	4,5	

EXERCISE 4.

$$\boldsymbol{MaxZ = 1,50x_1 + 2,50x_2 + 3,00x_3 + 4,50x_4}$$

$$2x_1 + 4x_2 + 3x_3 + 7x_4 \leq 100.000$$
$$3x_1 + 2x_2 + 3x_3 + 4x_4 \leq 50.000$$
$$2x_1 + 3x_2 + 2x_3 + 5x_4 \leq 60.000$$
$$x_1, x_2, x_3, x_4 \geq 0$$

$$\boldsymbol{MaxZ = 1,50x_1 + 2,50x_2 + 3,00x_3 + 4,50x_4 + 0x_5 + 0x_6 + 0x_7}$$

$$2x_1 + 4x_2 + 3x_3 + 7x_4 + x_5 = 100.000$$
$$3x_1 + 2x_2 + 3x_3 + 4x_4 + x_6 = 50.000$$
$$2x_1 + 3x_2 + 2x_3 + 5x_4 + x_7 = 60.000$$
$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$$

x_1: PA product quantity

x_2: PB product quantity

x_3: PC product quantity

x_4: PD product quantity

x_5: unused time for assembling (min)

x_6: unused time for polishing (min)

x_7: unused time for packaging (min)

EXERCISE 5

EXERCISE 5.

- The confectioner produces three **types of pudding: rice, tapioca and vanilla**. He has 108 units of milk, 150 units of sugar and 84 eggs available daily.
- The recipe for a **rice pudding** bowl requires 15 units of milk, 15 units of sugar and 9 eggs, and 24 portions of the prepared quantity can be served.
- The recipe for a **tapioca pudding** bowl requires 12 units of milk, 15 units of sugar and 9 eggs, and 18 portions of the prepared quantity can be served.
- The formula for one bowl of **vanilla pudding** requires 6 units of milk, 15 units of sugar and 6 eggs, and 12 portions of the prepared quantity can be served.
- Because of the demand of their customers, the confectioner must mix at least two bowls of each type of pudding daily. How many bowls of a type of pudding should be mixed if the confectioner wants to **produce the highest number of portions daily**?

a) Mathematically formulate this problem of linear programming.

b) Write the standard form of this problem and explain the meaning of structural and slack variables.

EXERCISE 5.

Bowls:	Rice pudding	Tapioca pudding	Vanilla puding	Constraint
Milk (units)	15	12	6	108
Sugar (units)	15	15	15	150
Eggs (number of)	9	9	6	84
Min. requirement Rice	1	0	0	2
Min. requirement Tapioca	0	1	0	2
Min. requirement Vanilla	0	0	1	2
Portions	24	18	12	

GENERAL MODEL:

Objective function:

$$\boldsymbol{MaxZ = 24x_1 + 18x_2 + 12x_3}$$

Constraints:

$$15x_1 + 12x_2 + 6x_3 \leq 108$$

$$15x_1 + 15x_2 + 15x_3 \leq 150$$

$$9x_1 + 9x_2 + 6x_3 \leq 84$$

$$x_1 \geq 2$$

$$x_2 \geq 2$$

$$x_3 \geq 2$$

Non-negativity condition:

$$x_1, x_2, x_3 \geq 0$$

EXERCISE 5.

Objective function:

$$\boldsymbol{MaxZ = 24x_1 + 18x_2 + 12x_3 + 0x_4 +}$$
$$\boldsymbol{+0x_5 + 0x_6 + 0x_7 + 0x_8 + 0x_9}$$

Constraints:

$$15x_1 + 12x_2 + 6x_3 + x_4 = 108$$
$$15x_1 + 15x_2 + 15x_3 + x_5 = 150$$
$$9x_1 + 9x_2 + 6x_3 + x_6 = 84$$
$$x_1 - x_7 = 2$$
$$x_2 - x_8 = 2$$
$$x_3 - x_9 = 2$$

Non-negativity condition:

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \geq 0$$

Interpretation of variables:

x_1: number of rice pudding bowls

x_2: number of tapioca pudding bowls

x_3: number of vanilla pudding bowls

x_4: unused ammount of milk (units)

x_5: unused ammount of sugar (units)

x_6: unused ammount of eggs (pieces)

x_7: exceeding the minimum required amount of rice pudding (bowls)

x_8: exceeding the minimum required amount of tapioca pudding (bowls)

x_9: exceeding the minimum required amount of vanilla pudding (bowls)

EXERCISE 6

EXERCISE 6.

- A dog breeder uses three types of food: **HA, HB and HC**, of which a mixture is made. The mixture must contain at least 240 units of carbohydrates, 200 units of protein and 120 units of fat.
- Food HA contains 20 carbohydrate units, 25 units of protein and 5 units of fat per unit; food HB contains 30 units of carbohydrates, 10 units of protein and 15 units of fat per unit while food HC contains 15 units of carbohydrates, 20 units of protein and 10 units of fat per unit.
- If the unit price of food HA is 3.00 HRK, food HB 6.00 HRK, and food HC 4.00 HRK, determine the ratio of food mix HA, HB and HC to **minimize the cost of dog food**.

a) Mathematically formulate this problem of linear programming.

b) Write the standard form of this problem and explain the meaning of structural and slack variables.

c) Suppose that an additional requirement is necessary: the amount of food of type HA should not be less than the amount of HC type of food. Complete the mathematical formulation.

EXERCISE 6.

	HA	HB	HC	Constraints
Carbohydrates	20	30	15	240
Protein	25	10	20	200
Fat	5	15	10	120
Price (kn)	3,00	6,00	4,00	← MIN!

General form:

Objective function:

$$\boldsymbol{MinW = 3x_1 + 6x_2 + 4x_3}$$

Constraints:

$$20x_1 + 30x_2 + 15x_3 \geq 240$$

$$25x_1 + 10x_2 + 20x_3 \geq 200$$

$$5x_1 + 15x_2 + 10x_3 \geq 120$$

Non-negativity condition:

$$x_1, x_2, x_3 \geq 0$$

EXERCISE 6.

CANONICAL FORM:

Objective function:

$$\boldsymbol{MinW = 3x_1 + 6x_2 + 4x_3 + 0x_4 + 0x_5 + 0x_6}$$

Constraints:

$$20x_1 + 30x_2 + 15x_3 - x_4 = 240$$

$$25x_1 + 10x_2 + 20x_3 - x_5 = 200$$

$$5x_1 + 15x_2 + 10x_3 - x_6 = 120$$

Non-negativity condition:

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Interpretation:

x_1: ammount of food HA (units)

x_2: ammount of food HB (units)

x_3: ammount of food HC (units)

x_4: exceeding the minimum required carbohydrates (units)

x_5: exceeding the minimum required protein (units)

x_6: exceeding the minimum required fat (units)

What to remember?

- Mathematical model, problem formulation
- Objective function, constraints, non-negativity
- Structural and slack variables

Anderson: Quantitative methods for business decisions –
Chapter 7: Introduction to linear programming

Thank you!

Questions?

Graphical solution of a linear programming model

Exercises 2.

International Business

ILO1	convert the practical problem of optimization into a mathematical statement as well as interpret results of applied quantitative models in business decision making

Graphical solution of a linear programming model

→ Solution, possible solution, optimal solution

→ **Solution** = each set of values of the variables that satisfies the equation system

→ **Possible solution** = the edge of a polygon; feasible, but not optimal

→ **Optimal solution** = the extreme point from a set of feasible solutions

Graphical solution of a linear programming model

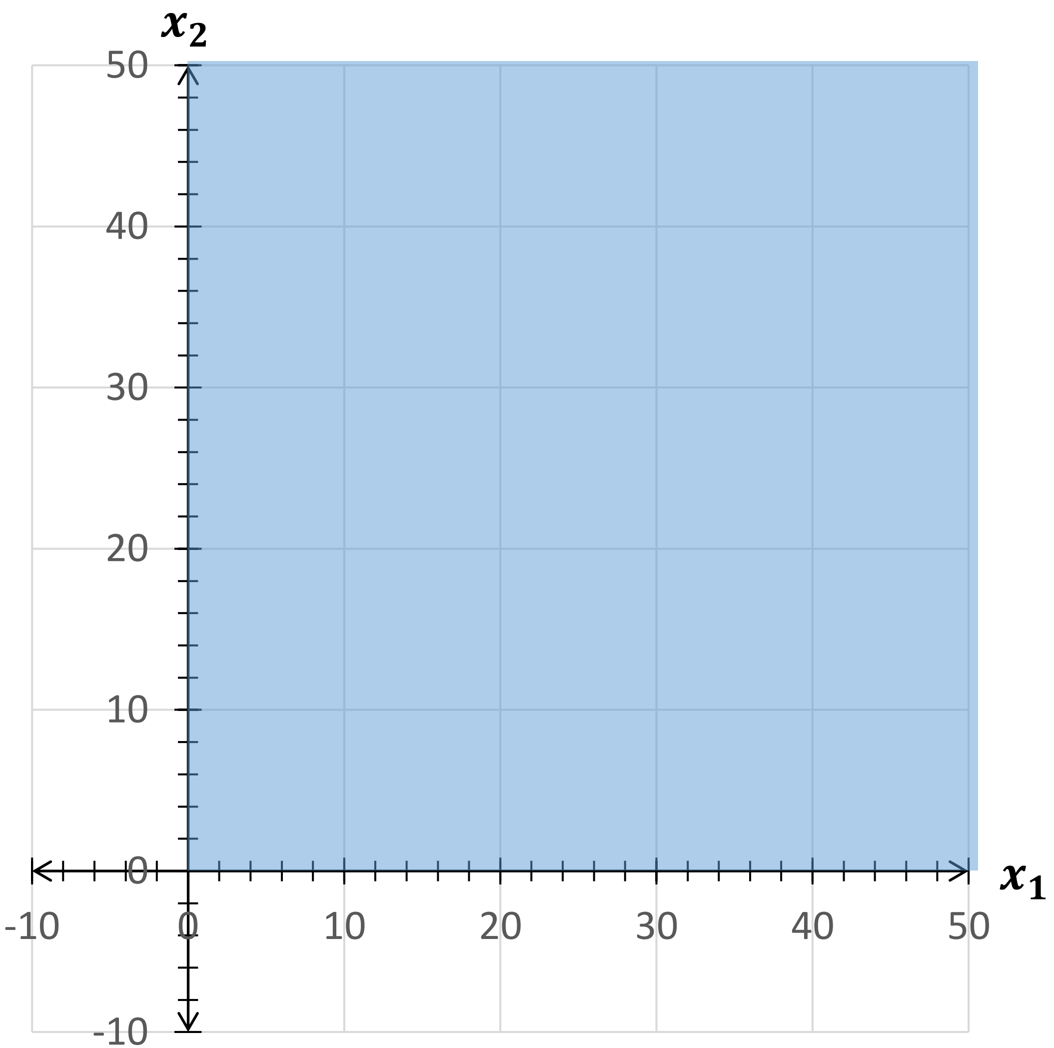
→ Feasible solution area, internal, borderline and extremne points

→ **feasible solution area** → determined by the boundary of the constraint system (including the boundary lines)

→ convex set

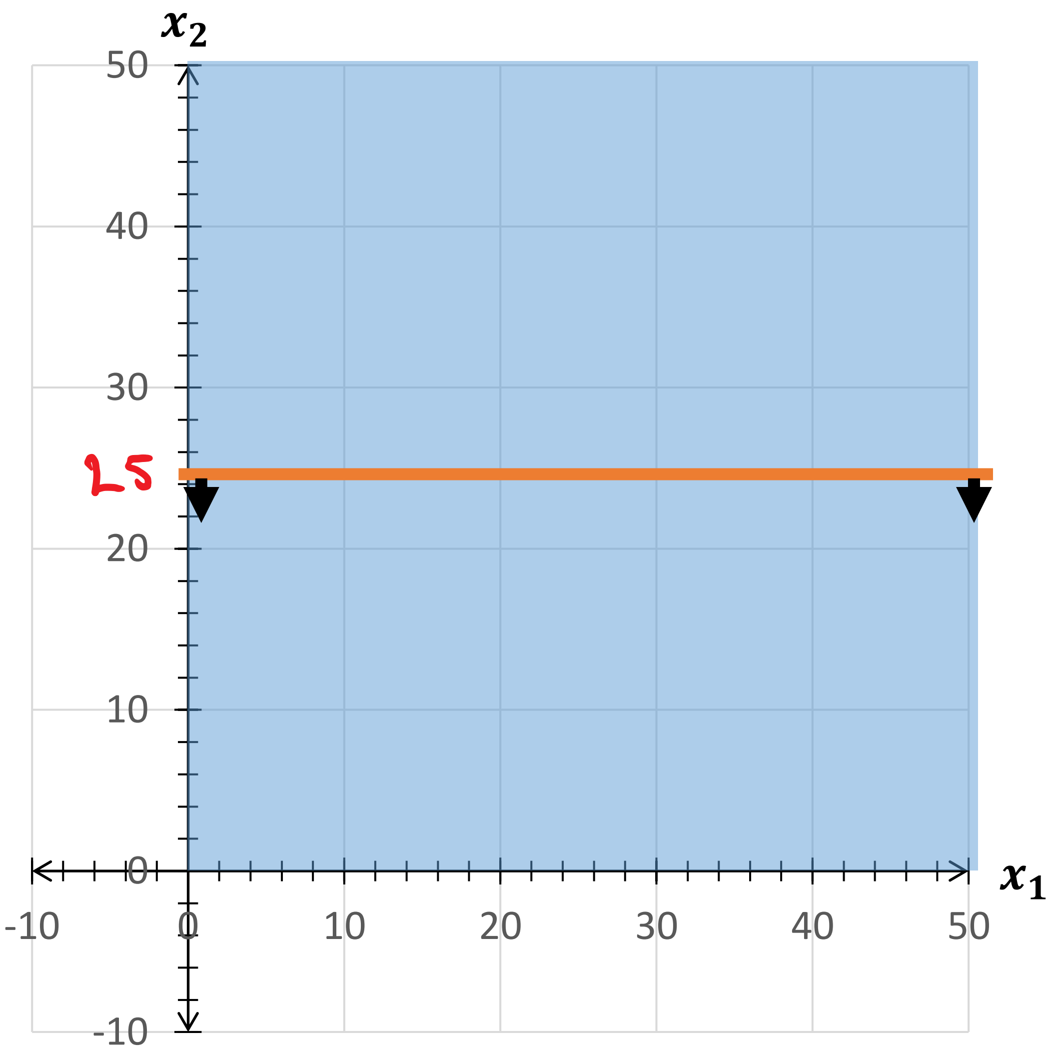
→vertices of the polygon = **extreme points**

Graphical solution of LP



Non-negativity condition:
$x_n \geq 0$

Graphical solution of LP



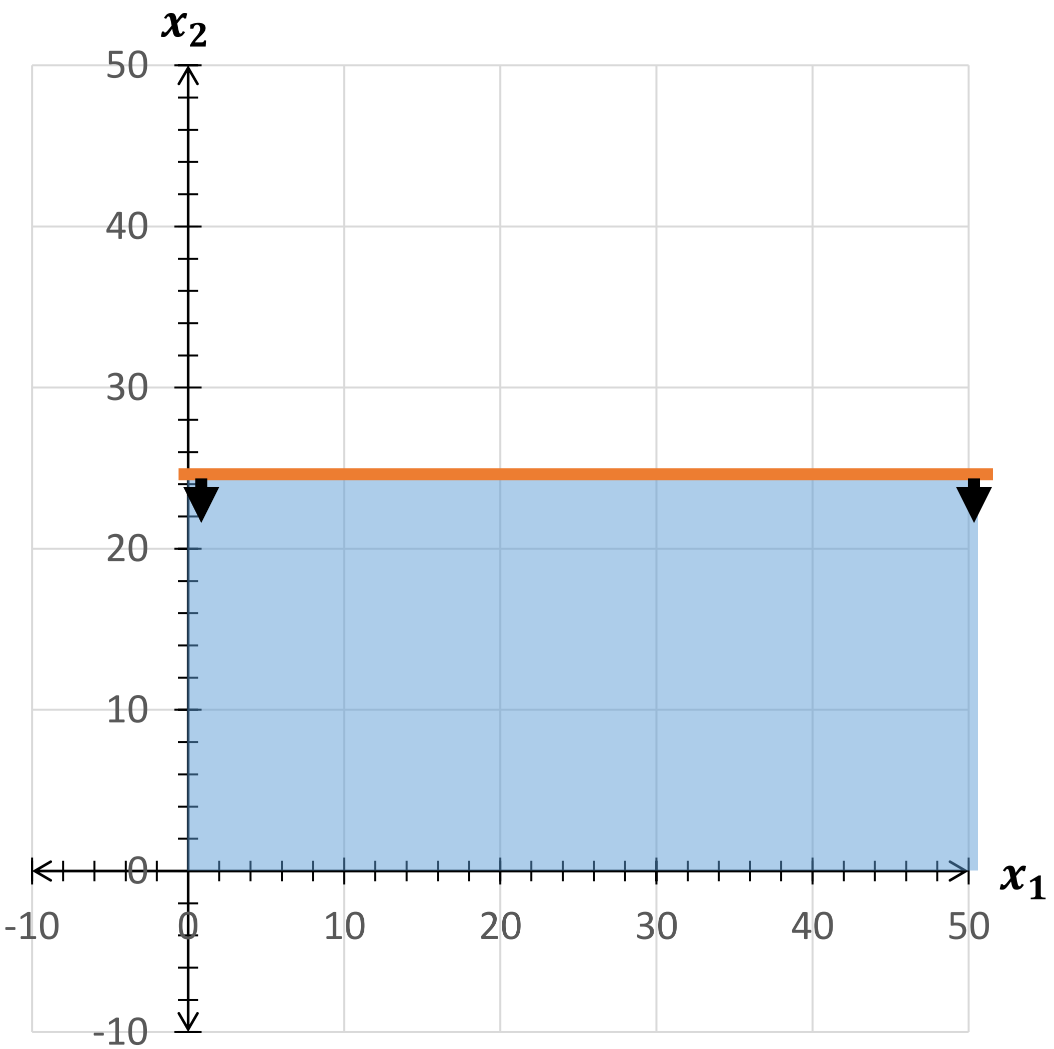
Non-negativity condition:

$x_n \geq 0$

Constraints:

$x_2 \leq 25$

Graphical solution of LP



Non-negativity condition:

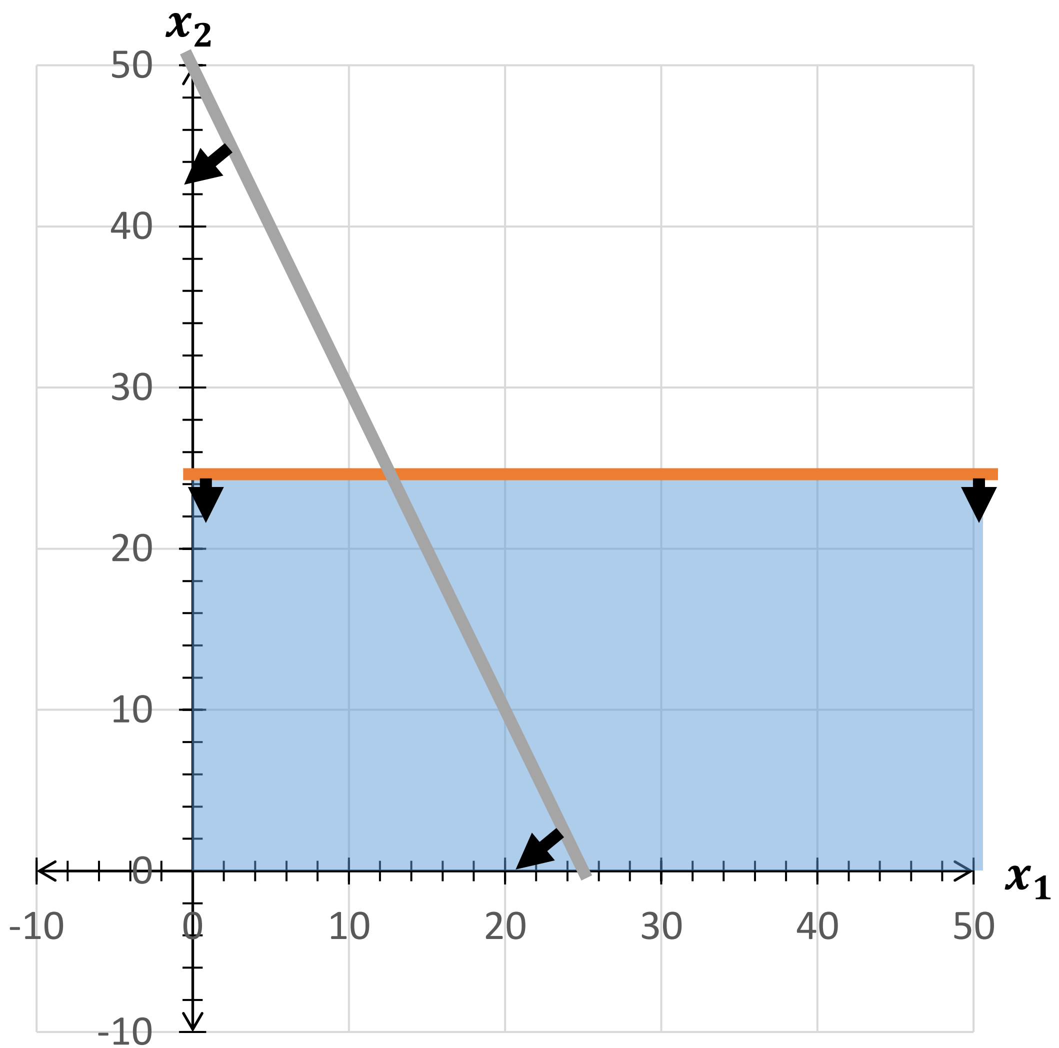
$x_n \geq 0$

Constraints:

$x_2 \leq 25$

$2x_1 + x_2 \leq 50$

Graphical solution of LP



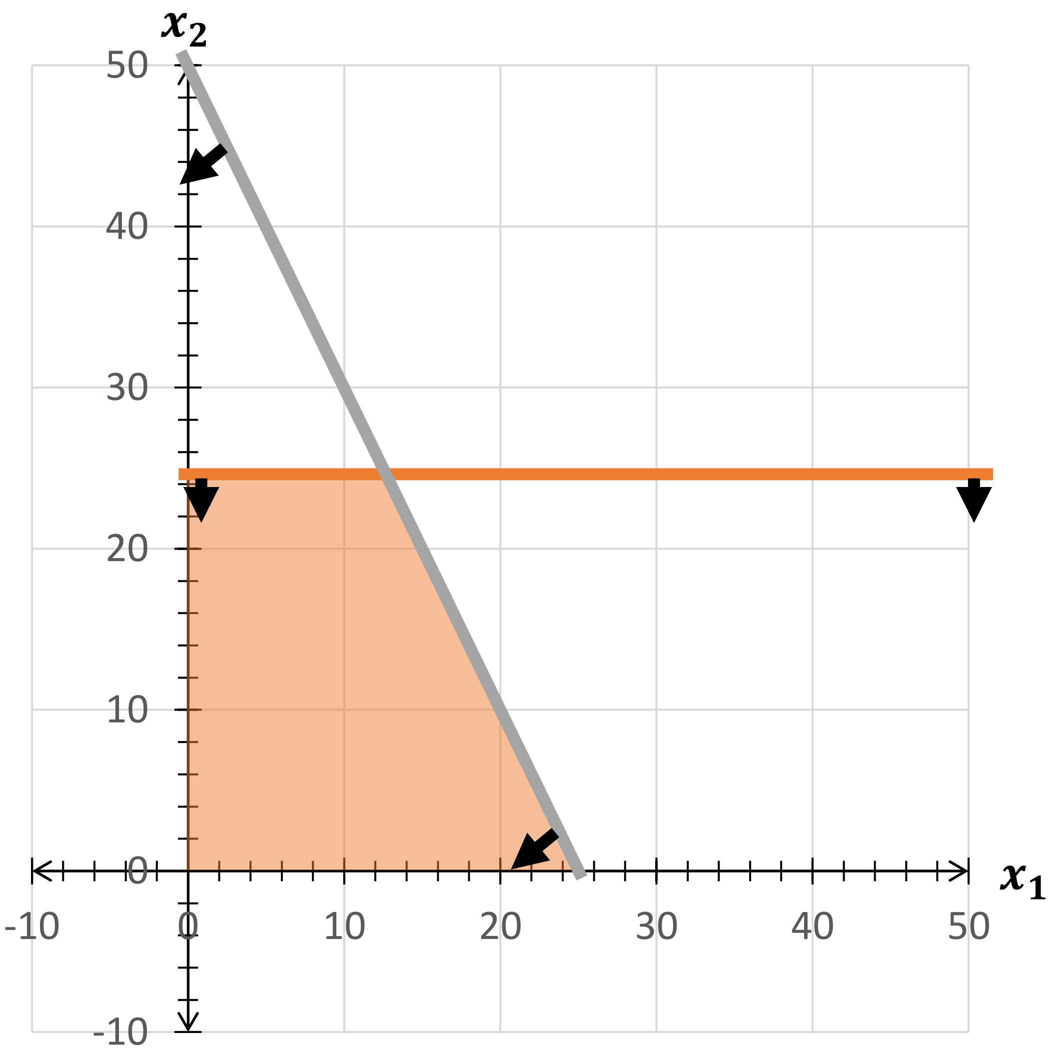
Non-negativity condition:
$x_n \geq 0$

Constraints:

$x_2 \leq 25$

$2x_1 + x_2 \leq 50$

Graphical solution of LP



Non-negativity condition:

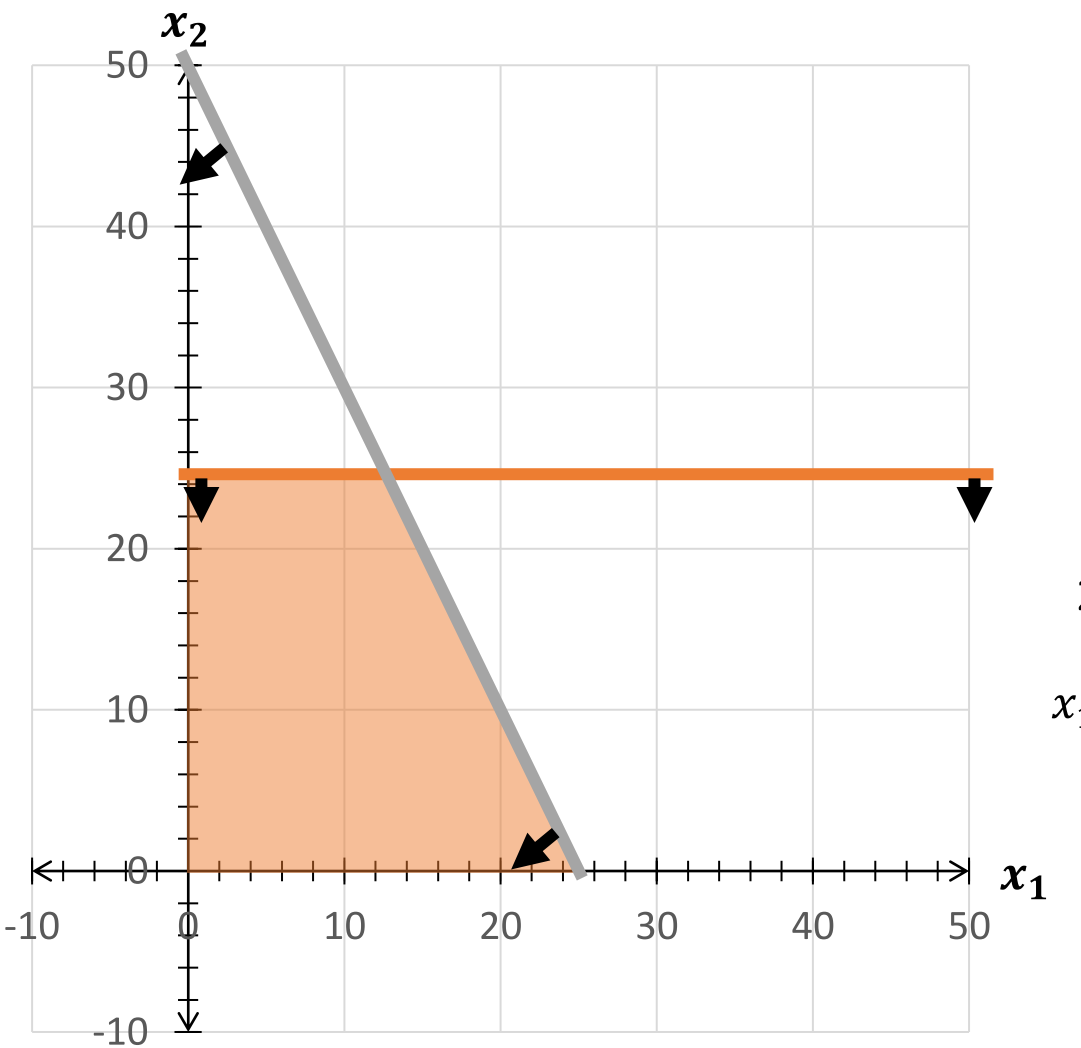
$x_n \geq 0$

Constraints:

$x_2 \leq 25$

$2x_1 + x_2 \leq 50$

Graphical solution of LP



Non-negativity condition:

$x_n \geq 0$

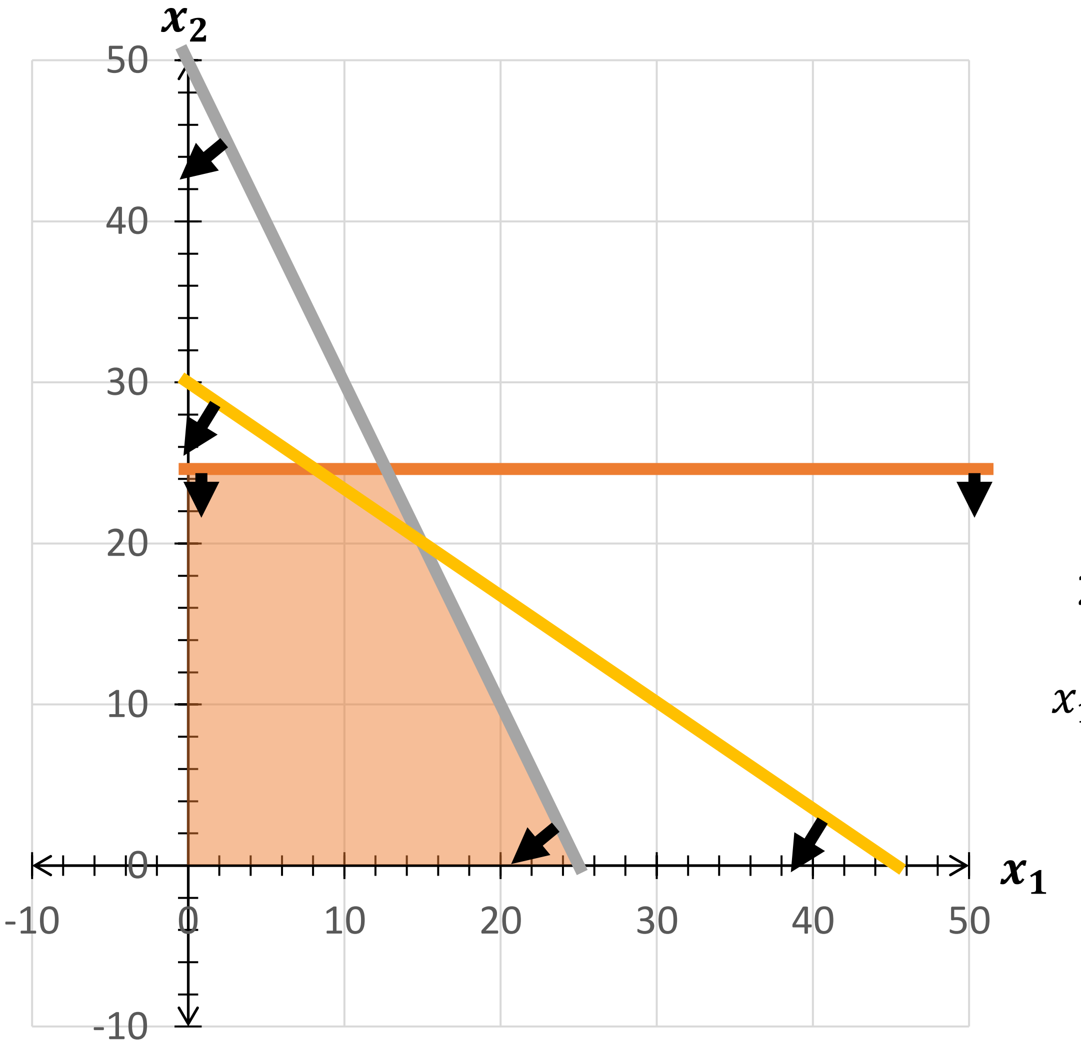
Constraints:

$x_2 \leq 25$

$2x_1 + x_2 \leq 50$

$x_1 + 1{,}5x_2 \leq 45$

Graphical solution of LP



Non-negativity condition:

$x_n \geq 0$

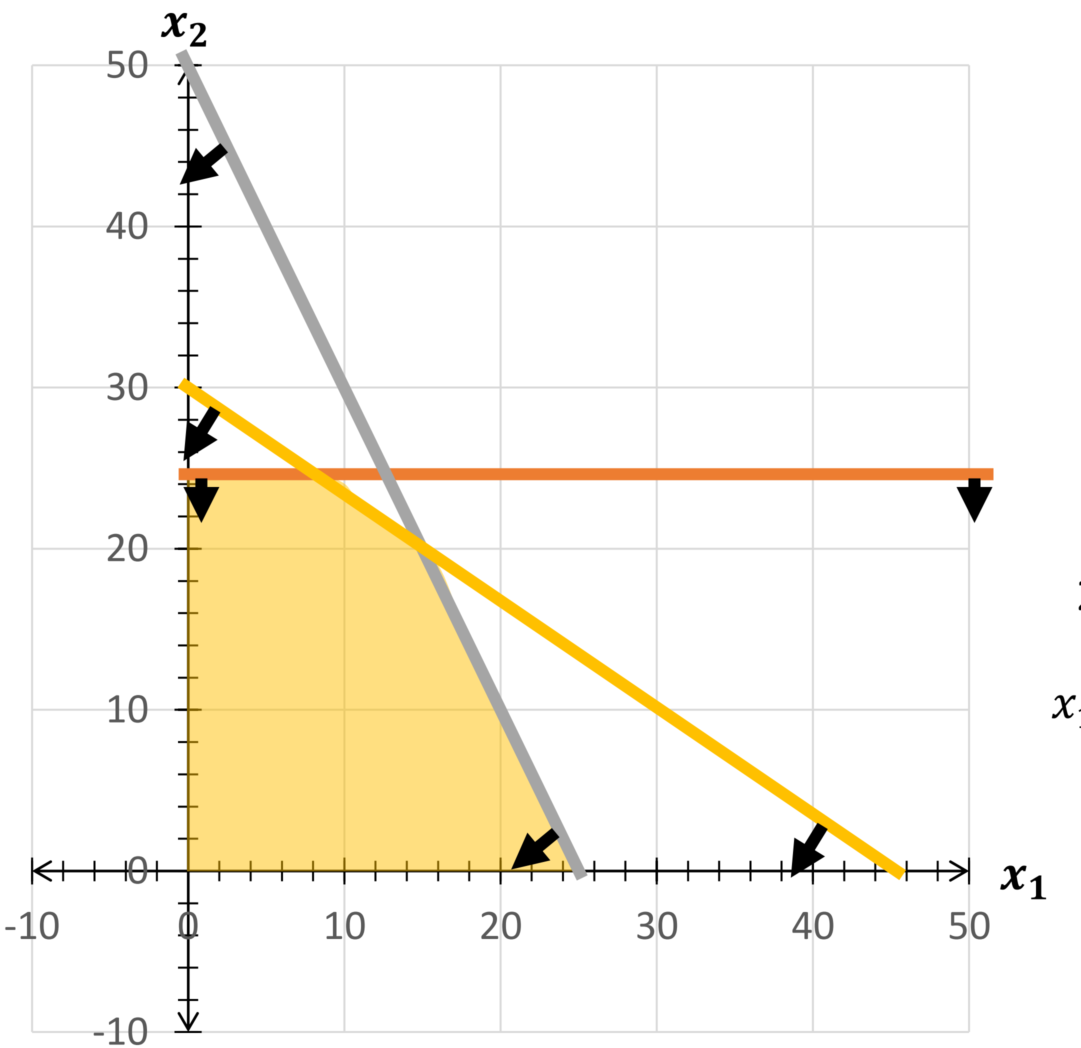
Constraints:

$x_2 \leq 20$

$2x_1 + x_2 \leq 50$

$x_1 + 1{,}5x_2 \leq 45$

Graphical solution of LP



Non-negativity condition:

$x_n \geq 0$

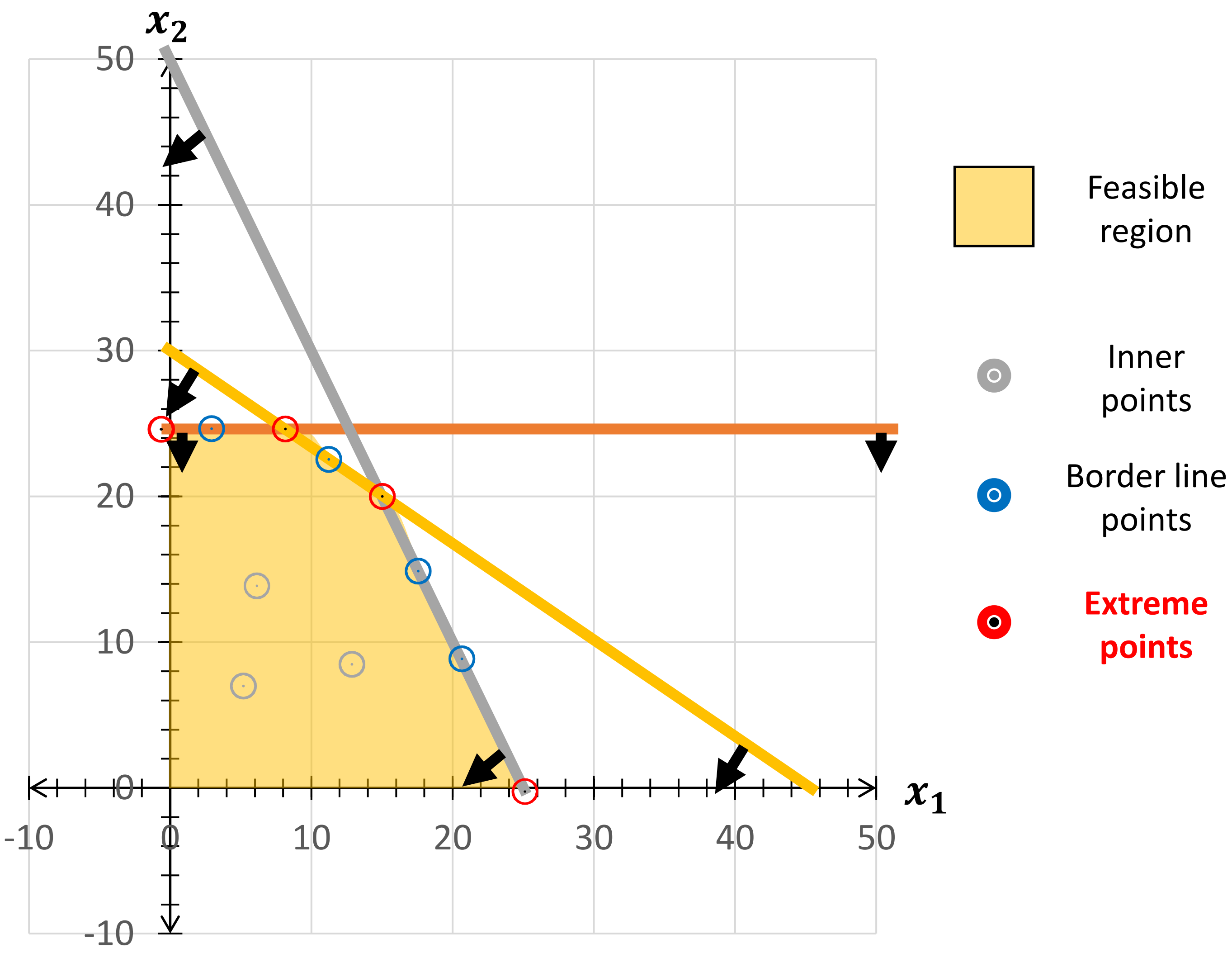
Constraints:

$x_2 \leq 20$

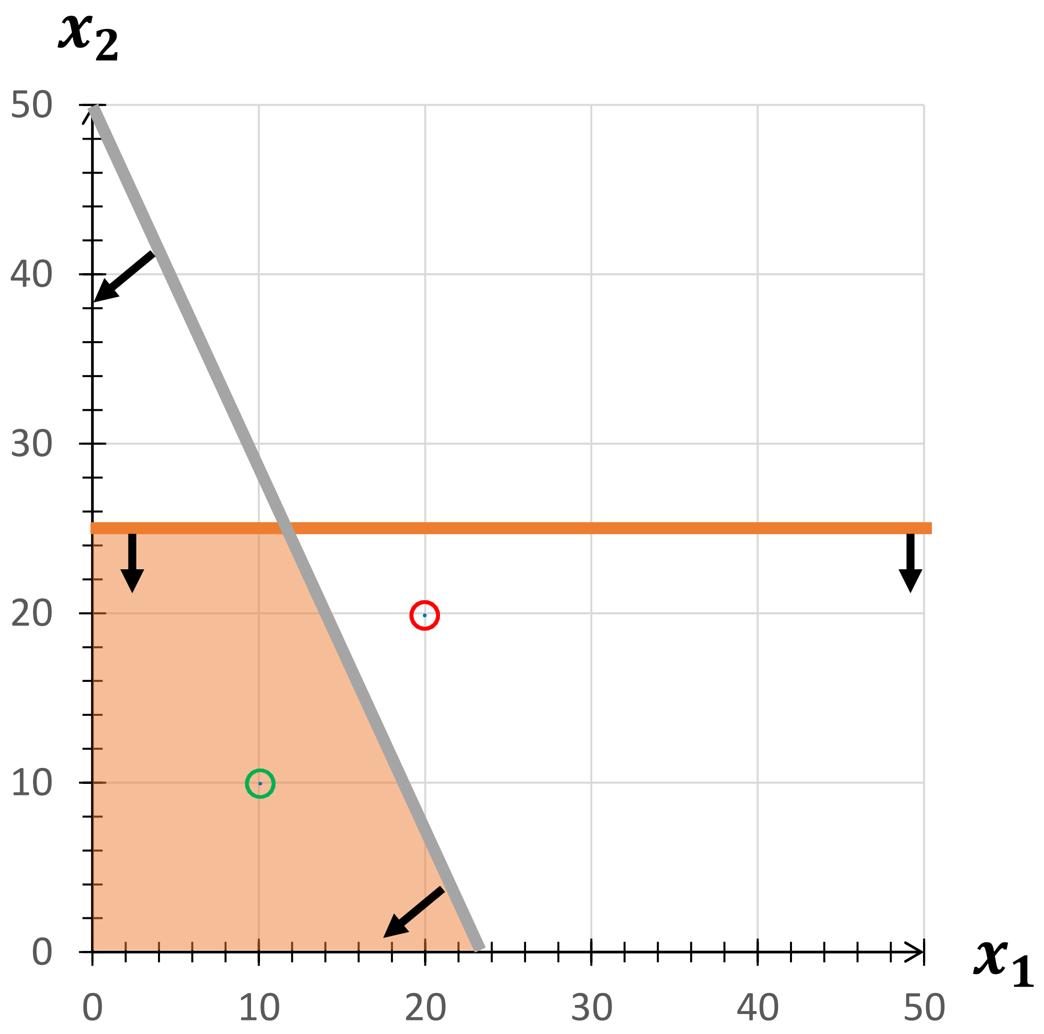
$2x_1 + x_2 \leq 50$

$x_1 + 1,5x_2 \leq 45$

Graphical solution of LP



Checking the constraints



Constraint:

$$2x_1 + x_2 \leq 50$$

Control point:

T (20, 20)

$$2 \times 20 + 20 \leq 50$$

$$\mathbf{60 \leq 50}$$

T (10, 10)

$$2 \times 10 + 10 \leq 50$$

$$\mathbf{30 \leq 50}$$

EXERCISE 1

EXERCISE 1.

- The company COOL sets up two electrical products: air conditioning units and special fans for a known customer.
- For the assembling of one of the air conditioning units, as well as for the assembling of one special fan, it takes 15 minutes, and the company has a day with 250 working hours for product assembling tasks.
- The time for quality control and packaging of the air conditioner is 9 minutes, and for the special fan unit 18 minutes, wherein the daily operating hours available for quality control and packaging are 210.
- Each special fan is fitted with one propeller, and the company's warehouse can provide 600 propellers per day. The customer asks that at least 20 % of all delivered products are special fans.
- If the company's profit is 15 EUR for the delivered air conditioner and 20 EUR for the special fan delivered, specify the daily production schedule of the air conditioning units or special fans that will give COOL the highest profit.

EXERCISE 1.

	Air conditioning unit	Special fan	Constraints
Assembling time	15 min	15 min	250 h
Quality control	9 min	18 min	210 h
Q of propellers		1	600 pieces
Min. Q of special fans		1	20 % of all
Profit	15 €	20 €	

EXERCISE 1.

- According to **task 2** of the previous exercises, the mathematical formulation of the linear programming problem is:

$$\boldsymbol{MaxZ = 15x_1 + 20x_2}$$

$$15x_1 + 15x_2 \leq 15.000$$

$$9x_1 + 18x_2 \leq 12.600$$

$$x_2 \leq 600$$

$$x_2 \geq 0{,}2(x_1 + x_2) \quad \rightarrow \quad 0{,}8x_2 - 0{,}2x_1 \geq 0$$

$$x_1, x_2 \geq 0$$

EXERCISE 1.

a) Draw a graphical set of possible solutions to this problem.

b) Determine the graph position of the objective function and find the optimal solution.

c) Interpret the obtained solution.

Constraint #1:

p_1 ...

$$15x_1 + 15x_2 \leq 15.000$$

* You need to determine 2 points on the graph's axis to draw the constraint line → if you quit producing one of the products (x = 0) you can direct all available resources to the production of the other.

$$x_1 = 0$$
$$15x_2 = 15.000 \quad / : 15$$
$$x_2 = 1.000$$

x_1 x_2

[0; 1.000]

* The values of x1 and x2 make a coordinate point on the graph axis

$$x_2 = 0$$
$$15x_1 = 15.000 \quad / : 15$$
$$x_1 = 1.000$$

x_1 x_2

[1.000; 0]

* Connect the two points and by doing so you will get the constraint line

Constraint #2:

p_2 ...

$$9x_1 + 18x_2 \leq 12.600$$

$$x_1 = 0$$
$$18x_2 = 12.600 \quad / : 18$$
$$x_2 = 700$$

x_1 x_2

[0; 700]

$$x_2 = 0$$
$$9x_1 = 12.600 \quad / : 9$$
$$x_1 = 1.400$$

x_1 x_2

[1.400; 0]

Constraint #3:

p_3 ...

$$x_2 \leq 600$$

$$x_2 = 600$$

$$\overset{x_1}{[0;}\ \overset{x_2}{600]}$$

* The third constraint concerns only the variable x2 → the constraint line will be perpendicular to the axis

Constraint #4:

p_4 ...

$$x_2 \geq 0{,}2(x_1 + x_2) \rightarrow$$

$$0{,}8x_2 - 0{,}2x_1 \geq 0 \;\; /:0{,}8 \rightarrow$$

$$x_2 \geq 0{,}25x_1$$

* In the case of this form of restrictions convert it so you get a coefficient of 1 with one of the variables

$$x = 0$$

$$x_1 = 0, \qquad x_2 = 0$$

$$x = 1.000$$

$$x_1 = 250, \qquad x_2 = 1.000$$

x_1	0	1.000
x_2	0	250

* With these forms of constraints, the starting point of the constraint is always in (0, 0)

- → After drawing all the constraints on the graph you will get a set of possible solutions (feasible solution area)
- → The feasible solution area is only that area that satisfies all constraints - a polygon that includes areas of all constraints
- → The last step is the **slope of the objective function**:
  - → You need to select a point from the feasible solution area (indicated by T in the graph)
  - → The coordinates of the selected point are then included in the objective function and you get a value from it→ that final value you can use with the same principle as with the constraints: you should find the coordinate points on each axis and connect them with a line [control step: the line you get by connecting the points on the axes must go through the point T you selected earlier]

$$\mathbf{T}\ (200, 300)$$

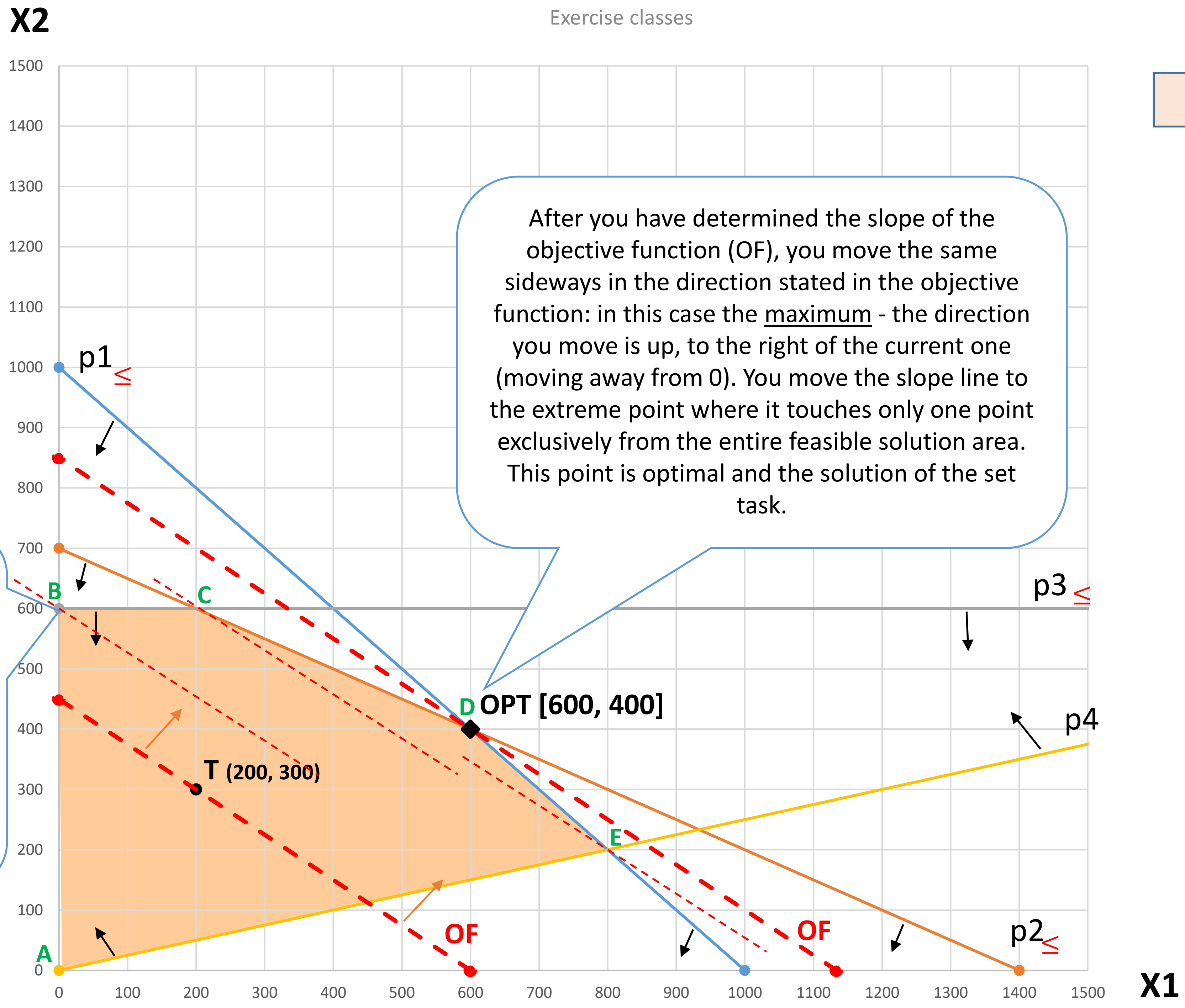
$$\boldsymbol{MaxZ = 15x_1 + 20x_2}$$

$$15 \times 200 + 20 \times 300 = 9.000$$

$$15x_1 + 20x_2 = 9.000$$

$$x_1 = 0, \quad x_2 = 450 \qquad [0; 450]$$

$$x_2 = 0, \quad x_1 = 600 \qquad [600; 0]$$



After you have determined the slope of the objective function (OF), you move the same sideways in the direction stated in the objective function: in this case the maximum - the direction you move is up, to the right of the current one (moving away from 0). You move the slope line to the extreme point where it touches only one point exclusively from the entire feasible solution area. This point is optimal and the solution of the set task.

The objective function goes through an extreme point, but also through other points within a set of possible solutions, and for that reason that point is not the optimal solution.

Feasible solution area – an area that meets all set restrictions

→

The optimal solution is the only point from the feasible solution area which touches the objective function line!

How to check the solution?

→ **set the extreme point values into the objective function and calculate:**

$$MaxZ = 15x_1 + 20x_2$$

(0, 0)	**(150, 600)**	**(200, 600)**	**(600, 400)**	**(1.000, 0)**
15 x 0 + 20 x 0	15 x 150 + 20 x 600	15 x 200 + 20 x 600	15 x 600 + 20 x 400	15 x 1.000 + 20 x 0
MaxZ = **0 €**	MaxZ = **14.250 €**	MaxZ = **15.000 €**	MaxZ = **17.000 €**	MaxZ = **15.000 €**

Proof that the point selected as optimal gives maximum profit.

Optimal solution

INTERPRETATION OF RESULTS

The *optimal point*, ie its coordinates, must be included in the overall model instead of the variables x1 and x2 to obtain the final results.

- **STRUCTURAL VARIABLES:**

- The optimal production is 600 pieces of air-conditioning units and 400 pieces of special fans.
  - OPT [600; 400]

- The maximum profit which can be made by selling 600 air-conditioning units and 400 special fans is 17.000 €.
  - $MaxZ = 15x_1 + 20x_2$; OPT [600; 400]

$$MaxZ = 15 \times 600 + 20 \times 400 = 17.000 \text{ €}$$

INTERPRETATION OF RESULTS

- **SLACK VARIABLES:**

- The available time for assembling is all used up.
  - $15x_1 + 15x_2 \leq 15.000;$ OPT $[600; 400]$

    $$15 \times 600 + 15 \times 400 = 15.000\ min$$

- The available time for quality control and packaging is all used up.
  - $9x_1 + 18x_2 \leq 12.600;$ OPT $[600; 400]$

    $9 \times 600 + 18 \times 400 = 12.600\ min$

INTERPRETATION OF RESULTS

- **SLACK VARIABLES:**
- There are 200 pieces of propellers left unused on stock.
  - $x_2 \leq 600$; OPT [600; 400]

$$400 < 600\ propellers$$

- The customers delivery condition is surpassed by 200 pieces of special fans.
  - $x_2 \geq 0{,}2(x_1 + x_2)$; OPT [600; 400]

$$400 > 0{,}2(600 + 400)$$

EXERCISE 2

EXERCISE 2.

- The company manufactures two types of products, Pro1 and Pro2, on two different machines, S001 and S002.
- For Pro1 production, 1 hour of machine S001 and 0.5 hours of machine S002 work is required, while Pro2 requires 1 hour of machine S001 and 1.5 hours of machine S002 work. The available daily capacity of the S001 machine is 16 hours, and the S002 machine is 12 hours.
- In one Pro1 product unit are 2 kilograms of MI1 material and 1 kg of MI2 material incorporated, while 1 kilogram of MI1 material is incorporated into product Pro2. 20 kg of MI1 material and 8 kg of MI2 material are on stock at the warehouse.
- The profit per product Pro1 amounts to 120.00 HRK, and per product Pro2 80.00 HRK, whereby the buyer requests from the manufacturer that the quantity of product Pro1 is at least 20% of the quantity of product Pro2. Determine the daily production schedule of P1 and P2 products that will maximize company profits.

EXERCISE 2.

	Pro1	Pro2	Constraint
Machine S001	1 h	1 h	16 h
Machine S002	0,5 h	1,5 h	12 h
Material MI1	2 kg	1 kg	20 kg
Material MI2	1 kg		8 kg
Customer requir.	1		≥ 20 % Pro 2
Dobit	120,00 kn	80,00 kn	

EXERCISE 2.

- According to **task 3** of the previous exercises, the mathematical formulation of the linear programming problem is:

$$\boldsymbol{MaxZ = 120x_1 + 80x_2}$$

$$x_1 + x_2 \leq 16$$
$$0{,}5x_1 + 1{,}5x_2 \leq 12$$
$$2x_1 + 1x_2 \leq 20$$
$$x_1 \leq 8$$
$$x_1 \geq 0{,}2x_2$$
$$x_1, x_2 \geq 0$$

x_1: Pro1 product quantity

x_2: Pro2 product quantity

x_3: unused machine hours S001 (h)

x_4: unused machine hours S002 (h)

x_5: unused amount of MI1 material (kg)

x_6: unused amount of MI2 material (kg)

x_7: the amount of product Pro1 delivered exeeding the buyers requirement

EXERCISE 2.

a) Draw a graphical set of possible solutions to this problem.

b) Determine the graph position of the objective function and find the optimal solution.

c) Interpret the obtained solution.

Constraint #1:

p_1 ...

$x_1 + x_2 \leq 16$

* You need to determine 2 points on the graph's axis to draw the constraint line → if you quit producing one of the products (x = 0) you can direct all available resources to the production of the other.

$x_1 = 0$

$x_2 = 16$

x_1 x_2

[0; 16]

* The values of x1 and x2 make a coordinate point on the graph axis

$x_2 = 0$

$x_1 = 16$

x_1 x_2

[16; 0]

* Connect the two points and by doing so you will get the constraint line

Constraint #2:

p_2 ...

$0{,}5x_1 + 1{,}5x_2 \leq 12$

$x_1 = 0$

$1{,}5x_2 = 12 \quad / : 1{,}5$

$x_2 = 8$

x_1 x_2

[0; 8]

$x_2 = 0$

$0{,}5x_1 = 12 \quad / : 0{,}5$

$x_1 = 24$

x_1 x_2

[24; 0]

Constraint #3:

p_3 …

$$2x_1 + 1x_2 \leq 20$$

$$x_1 = 0$$
$$x_2 = 20$$

x_1 x_2

[0; 20]

$$x_2 = 0$$
$$2x_1 = 20 \qquad / : 2$$

x_1 x_2

[20; 0]

Constraint #4:

p_4 …

$$x_1 \leq 8$$
$$x_1 = 8$$

x_1 x_2

[0; 8]

* The fourth constraint concerns only the variable x1 → the constraint line will be perpendicular to the axis

Constraint #5:

p_5 …

$$x_1 \geq 0{,}2x_2$$

* In the case of this form of restrictions convert it so you get a coefficient of 1 with one of the variables

$$x = 0$$

$$x_1 = 0, \qquad x_2 = 0$$

$$x = 10$$

$$x_1 = 10, \qquad x_2 = 2$$

x_1	0	2
x_2	0	10

* With these forms of constraints, the starting point of the constraint is always in (0, 0)

→ After drawing all the constraints on the graph you will get a set of possible solutions (feasible solution area)
→ The feasible solution area is only that area that satisfies all constraints - a polygon that includes areas of all constraints
→ The last step is the **slope of the objective function**:
  → You need to select a point from the feasible solution area (indicated by T in the graph)
  → The coordinates of the selected point are then included in the objective function and you get a value from it→ that final value you can use with the same principle as with the constraints: you should find the coordinate points on each axis and connect them with a line [control step: the line you get by connecting the points on the axes must go through the point T you selected earlier]

$$\boldsymbol{MaxZ = 120x_1 + 80x_2}$$

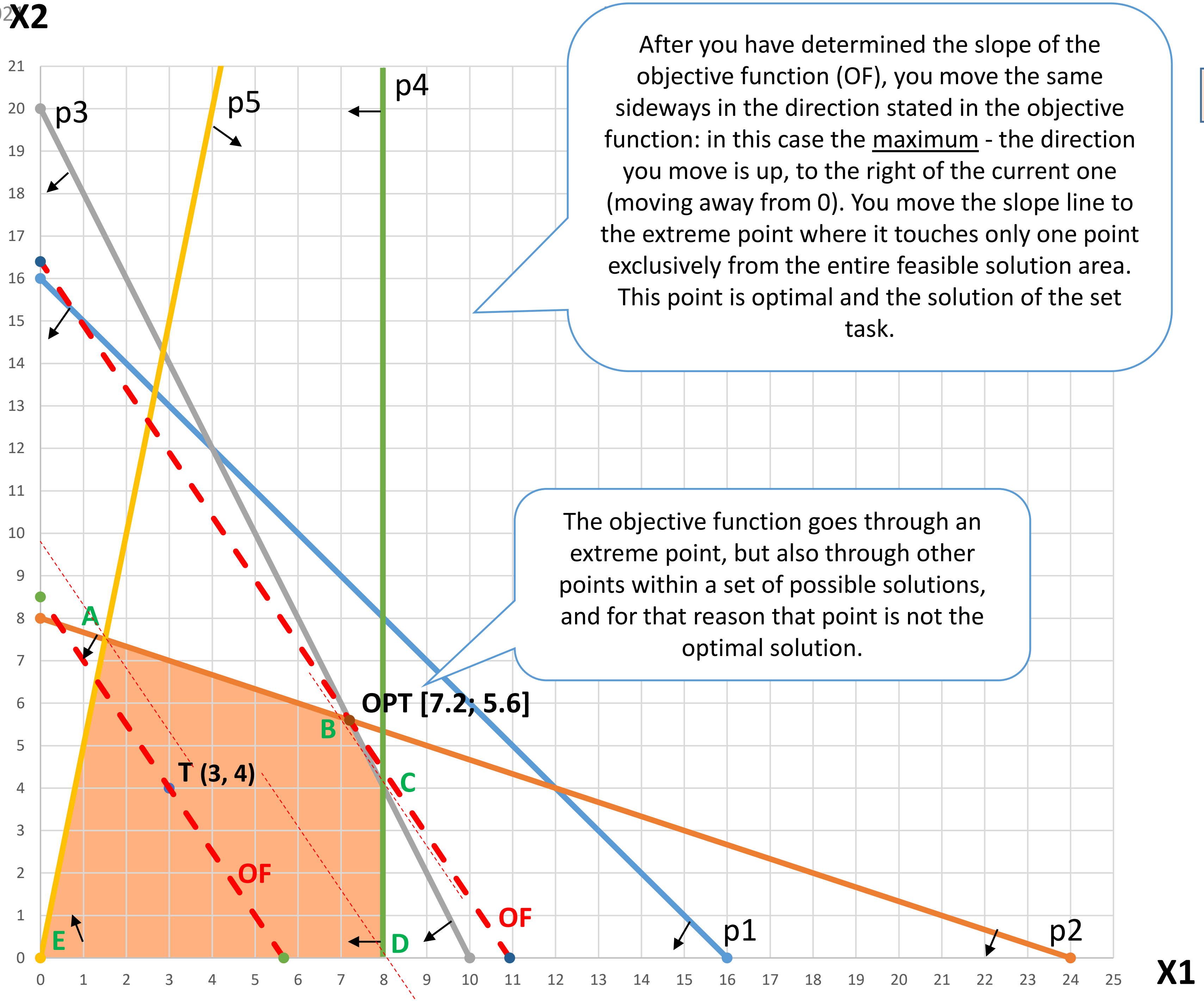
$\mathbf{T}\ (3, 4)$

$$120 \times 3 + 80 \times 4 = 680$$

$$120x_1 + 80x_2 = 680$$

$x_1 = 0, \ \ x_2 = 8{,}5$ [0; 8,5]

$x_2 = 0, \ \ x_1 = 5{,}67$ [5,67; 0]



Feasible solution area – an area that meets all set restrictions

→

The optimal solution is the only point from the feasible solution area which touches the objective function line!

INTERPRETATION OF RESULTS

The *optimal point*, ie its coordinates, must be included in the overall model instead of the variables x1 and x2 to obtain the final results.

- **STRUCTURAL VARIABLES:**

- The optimal production is 7,2 products Pro1 and 5,6 products Pro2.
  - OPT [7,5; 5,6]
- The maximum profit which can be made by selling 7,2 products Pro1 and 5,6 products Pro2 is 1.312 kn.
  - $MaxZ = 120x_1 + 80x_2$; OPT [7,2; 5,6]

$$MaxZ = 120 \times 7{,}2 + 80 \times 5{,}6 = 1.312\ kn$$

INTERPRETATION OF RESULTS

- **SLACK VARIABLES:**

- The available working hours of machine S001 are not used up entirely. There is an unused amount of 3,2 working hours.
  - $x_1 + x_2 \leq 16;$ OPT [7,2; 5,6]

$$1 \times 7{,}2 + 1 \times 5{,}6 = \mathbf{12{,}8} < \mathbf{16}\ h$$

- The available working hours of machine S002 are used up entirely.
  - $0{,}5x_1 + 1{,}5x_2 \leq 12;$ OPT [7,2; 5,6]

$$0{,}5 \times 7{,}2 + 1{,}5 \times 5{,}6 = \mathbf{12} = \mathbf{12}\ h$$

INTERPRETATION OF RESULTS

- **SLACK VARIABLES:**

- The available amount of material MI1 is all used up.
  - $2x_1 + 1x_2 \leq 20$; OPT [7,2; 5,6]

$$2 \times 7{,}2 + 1 \times 5{,}6 = 20 = 20$$

- The available amount of material MI2 is not used up entirely. There is an unused amount of 0,8 kg MI2 material on stock.
  - $x_1 \leq 8$; OPT [7,2; 5,6]

$$7{,}2 < 8$$

INTERPRETATION OF RESULTS

- **SLACK VARIABLES:**

- The customers requirement is surpassed, 6,08 units of product Pro1 above the minimum requirement were delivered.
  - $x_1 \geq 0{,}2x_2$; OPT [7,2; 5,6]

$$7{,}2 > 0{,}2 \times 5{,}6 \quad \rightarrow \quad 7{,}2 > 1{,}12$$

EXERCISE 3

EXERCISE 3.

- The following objective function is given:

$$\boldsymbol{MinW = 60x_1 + 24x_2}$$

- Determine the feasible area and minimum of the objective function with the following limitations:

$$3x_1 + 9x_2 \geq 45$$
$$3x_1 + 3x_2 \geq 30$$
$$3x_1 \geq 12$$
$$x_2 \geq 0$$

$$x_1, x_2 \geq 0$$

a) Draw a graphical set of possible solutions to this problem.

b) Determine the graph position of the objective function and find the optimal solution.

p_1 ...

$$3x_1 + 9x_2 \geq 45$$

$x_1 = 0, \quad x_2 = 5$

$x_2 = 0, \quad x_1 = 15$

$x_1 \quad x_2 \quad x_1 \quad x_2$

[15; 0] i [0; 5]

p_2 ...

$$3x_1 + 3x_2 \geq 30$$

$x_1 = 0, \quad x_2 = 10$

$x_2 = 0, \quad x_1 = 10$

[10; 0] i [0; 10]

p_3 ...

$$3x_1 \geq 12$$

$x_2 = 4$

[0; 4]

p_4 ...

$$x_2 \geq 0$$

$x_2 = 0$

[0; 0]

We do not need to draw the line of the constraint, because it overlaps with the axis x2 of the quadrant → already covered by the model (non-negativity).

$\mathbf{T}\ (8, 6)$

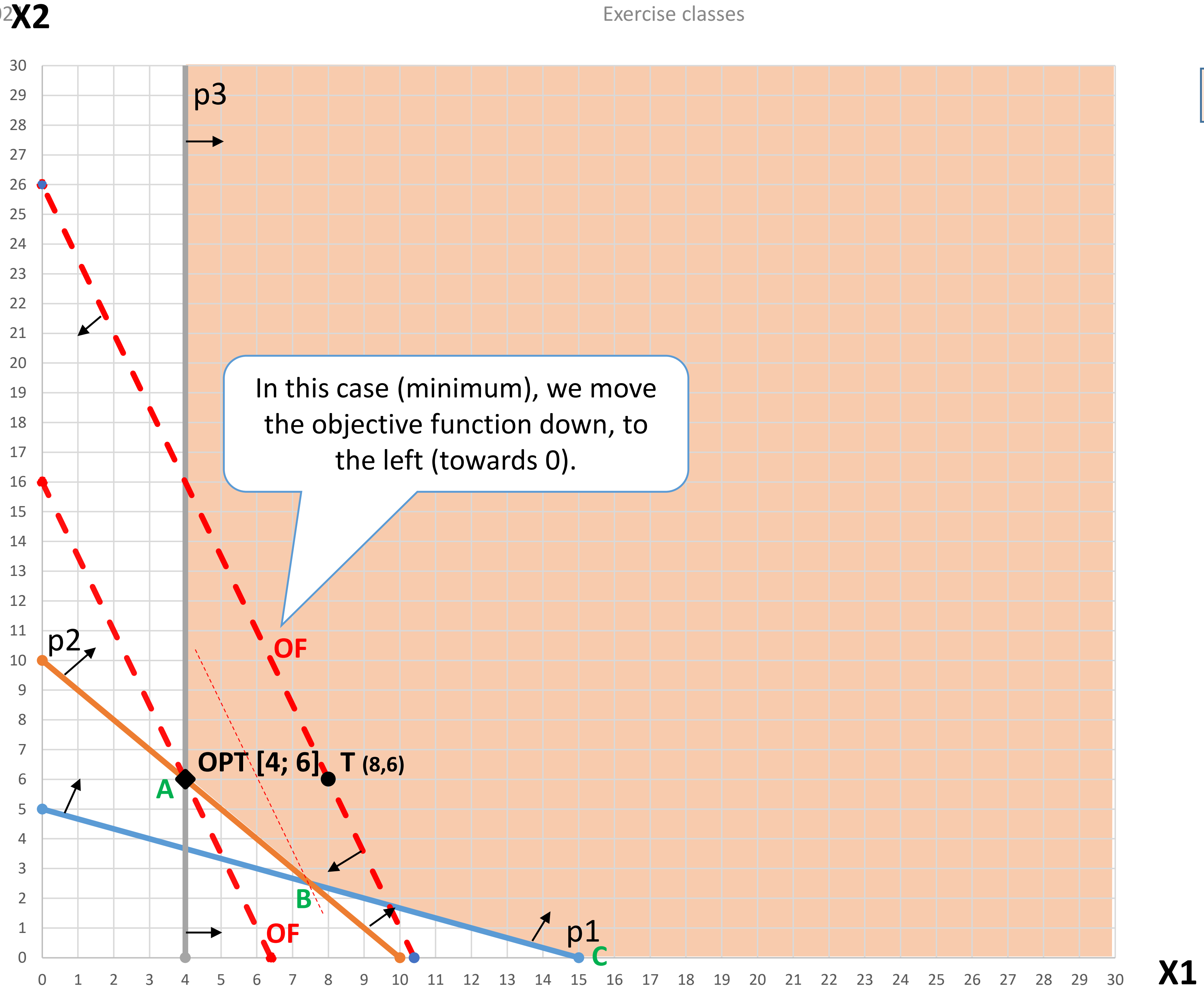
$$\boldsymbol{Min\, W = 60x_1 + 24x_2}$$

$$60 \times 8 + 24 \times 6 = 624$$

$$60x_1 + 24x_2 = 624$$

$x_1 = 0, \ \ x_2 = 26$ [0; 26]

$x_2 = 0, \ \ x_1 = 10{,}4$ [10,4; 0]



Feasible solution area – an area that meets all set restrictions

→

The optimal solution is the only point from the feasible solution area which touches the objective function line!

EXERCISE 4

EXERCISE 4.

- The asylum for cats needs two types of food: K1 and K2. Each preparation mixture of foods must contain at least 260 grams of carbohydrates, 220 grams of protein and 120 grams of fat.
- Food K1 contains 20 grams of carbohydrates, 13,75 grams of protein and 5 grams of fat per unit while food K2 contains 15 grams of carbohydrates, 20 grams of protein and 20 grams of fat per unit.
- If the unit price of food K1 is 13,00 HRK, and the food K2 16,00 HRK, determine the ratio of food dosage K1 and K2 which will minimize the cost of cats nutrition.

EXERCISE 4.

a) Mathematically formulate this problem of linear programming.

b) Write the standard form of this problem and explain the meaning of structural and slack variables.

c) Draw a graphical set of possible solutions to this problem.

d) Determine the graph position of the objective function and find the optimal solution.

e) Interpret the obtained solution.

ZADATAK 4.

	K1	K2	Constraint
Carbohydrates	20 g	15 g	260 g
Protein	13,75 g	20 g	220 g
Fat	5 g	20 g	120 g
Cost	13,00 HRK	16,00 HRK	

ZADATAK 2.

General form:

$$\boldsymbol{Min\, W = 13x_1 + 16x_2}$$
$$20x_1 + 15x_2 \geq 260$$
$$13{,}75x_1 + 20x_2 \geq 220$$
$$5x_1 + 20x_2 \geq 120$$
$$x_1, x_2 \geq 0$$

Standard form:

$$\boldsymbol{Min\, W = 13x_1 + 16x_2 + 0x_3 + 0x_4 + 0x_5}$$
$$20x_1 + 15x_2 - x_3 = 260$$
$$13{,}75x_1 + 20x_2 - x_4 = 220$$
$$5x_1 + 20x_2 - x_5 = 120$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

x_1: quantity of food K1

x_2: quantity of food K2

x_3: exceeding the minimum required amount of carbohydrates (in grams)

x_4: exceeding the minimum required amount of protein (in grams)

x_5: exceeding the minimum required amount of fat (in grams)

p_1 ...

$$20x_1 + 15x_2 \geq 260$$

$$x_1 = 0, \qquad x_2 = 17{,}3$$

$$x_2 = 0, \qquad x_1 = 13$$

[13; 0] i [0; 17,3]

$x_1 \quad x_2 \quad x_1 \quad x_2$

p_2 ...

$$13{,}75x_1 + 20x_2 \geq 220$$

$$x_1 = 0, \qquad x_2 = 11$$

$$x_2 = 0, \qquad x_1 = 16$$

[16; 0] i [0; 11]

p_3 ...

$$5x_1 + 20x_2 \geq 120$$

$$x_1 = 0, \qquad x_2 = 6$$

$$x_2 = 0, \qquad x_1 = 24$$

[24; 0] i [0; 6]

$$\mathbf{T}\ (10, 10)$$

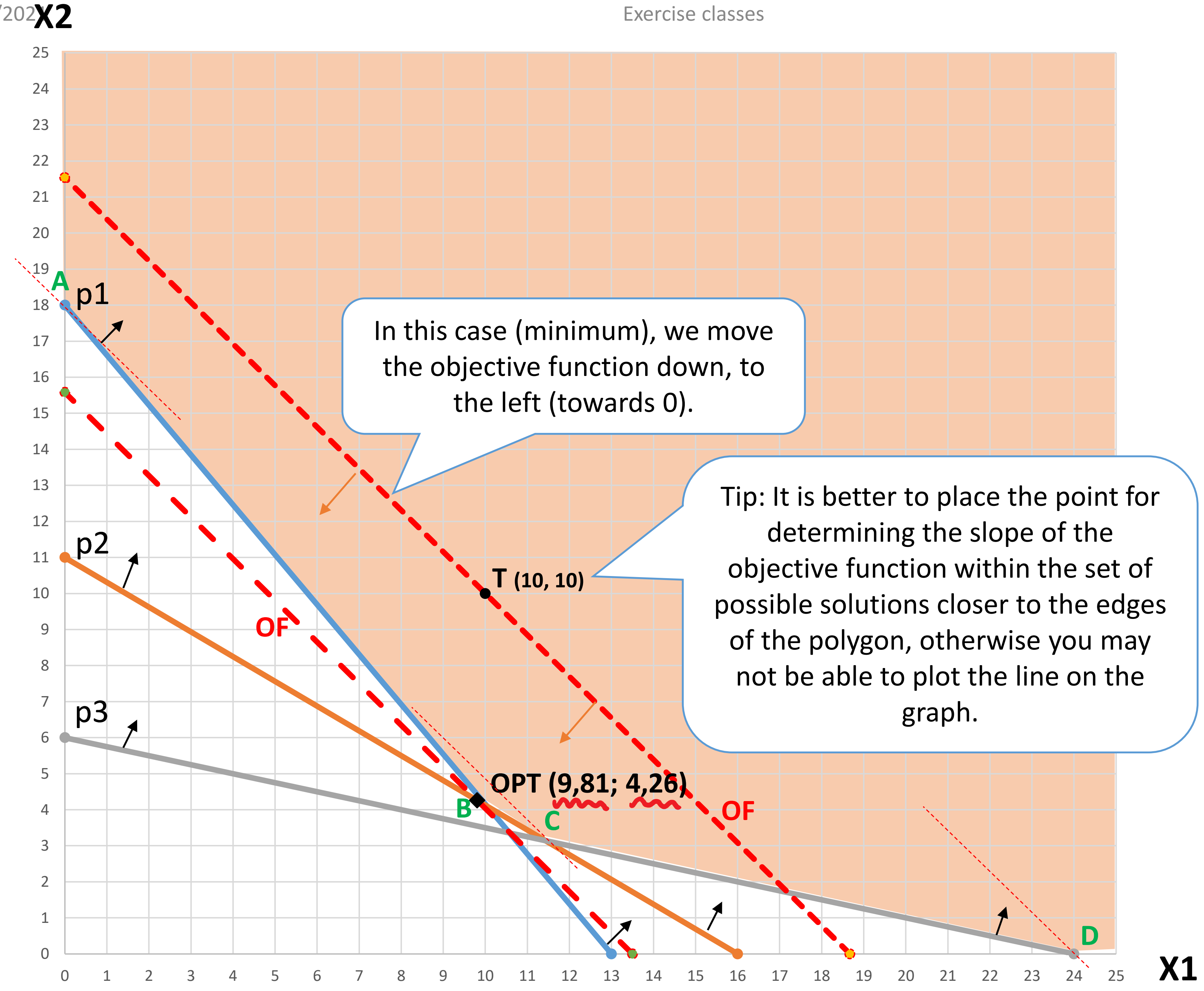
$$\boldsymbol{MinW = 13x_1 + 16x_2}$$

$$13 \times 10 + 16 \times 10 = 290$$

$$13x_1 + 16x_2 = 290$$

$x_1 = 0, \ \ x_2 = 18{,}1$ [0; 18,1]

$x_2 = 0, \ \ x_1 = 22{,}3$ [22,3; 0]



Feasible solution area – an area that meets all set restrictions

→

The optimal solution is the only point from the feasible solution area which touches the objective function line!

INTERPRETATION OF RESULTS

The *optimal point*, ie its coordinates, must be included in the overall model instead of the variables x1 and x2 to obtain the final results.

- **STRUCTURAL VARIABLES:**

- The optimal amount of food used is 9,81 units of food K1 and 4,26 units of food K2.
  - OPT [9,81; 4,26]
- The minimum cost which can be achieved by purchasing 9,81 units of food K1 and 4,26 units of food K2 is 195,69 kn.
  - $MinW = 13x_1 + 16x_2$; [9,81; 4,26]

$$MinW = 13 \times 9{,}81 + 16 \times 4{,}26 = 195{,}69\ kn$$

INTERPRETATION OF RESULTS

- **SLACK VARIABLES:**

- We have exceeded the minimum required amount of carbohydrates by 0,1 gram.
  - $20x_1 + 15x_2 \geq 260$; OPT [9,81; 4,26]

$$20 \times 9{,}81 + 15 \times 4{,}26 = \mathbf{260,1} > \mathbf{260}\ g$$

- We have exceeded the minimum required amount of protein by 0,09 gram.
  - $13{,}75x_1 + 20x_2 \geq 220$; OPT [9,81; 4,26]

$$13{,}75 \times 9{,}81 + 20 \times 4{,}26 = \mathbf{220,09} > \mathbf{220}\ g$$

- We have exceeded the minimum required amount of fat by 14,25 gram.
  - $5x_1 + 20x_2 \geq 120$; OPT [9,81; 4,26]

$$5 \times 9{,}81 + 20 \times 4{,}26 = \mathbf{134,25} > \mathbf{120}\ g$$

What to remember?

- Feasible solution area, determination of constraints, slope of the objective function
- Solution, possible solution, optimal solution
- Convex set, extreme points

Anderson: Quantitative methods for business decisions – Chapter 7: Introduction to linear programming

Thank you!

Questions?

Using MS Excel in linear programming

Exercises 3.

International Business

ILO1	convert the practical problem of optimization into a mathematical statement as well as interpret results of applied quantitative models in business decision making

Tools for solving linear models

- more complex multi-variable tasks need to be addressed through available software tools

- Microsoft Excel

→ you need to enable an add-in in the program: Solver

→ *Activation procedure*: File → Options → Add-Ins → Manage: Excel Add-ins → Go… → ✓ Anslysis ToolPak; ✓ Solver Add-in → OK

→ https://support.office.com/en-us/article/load-the-solver-add-in-in-excel-612926fc-d53b-46b4-872c-e24772f078ca

→ https://www.youtube.com/watch?v=W6tIS4JZ5J0

→ Instructions for solving linear programming problems using the Solver tool

EXERCISE 1

Exercise 1.

- The pension fund is considering investing in securities. There is a total investment fund of $ 1.000.000,00 available. They are considering to invest into shares at a price of $ 1.500,00 with an annual yield of 8 % and bonds at a price of $ 1.000,00 with an annual yield of 5 %. Next year it is necessary to make a profit on the securities of at least $ 40.000,00. Due to the risk of investment, it was decided not to invest more than 30 % of the available funds in shares. It is therefore necessary to place at least part of the funds into bonds, but taking into account that each individual investor cannot buy less than 500 bonds.
- An optimal investment strategy (number of shares and bonds) should be determined if the goal is to maximize total annual earnings, taking into account all set limits.

1) Writing the program

	Shares	**Bonds**	Constraints

„There is a total investment fund of $ 1,000,000.00 available. They are considering to invest into shares at a price of $ 1,500.00 and bonds at a price of $ 1,000.00."

Budget	1.500	1.000	**1.000.000**

„ Next year it is necessary to make a profit on the securities of at least $ 40,000.00. "

„**...** shares at a price of $ 1,500.00 with an annual yield of 8 % and bonds at a price of $ 1,000.00 with an annual yield of 5 %."

Earnings t+1	120	50	**40.000**

1.500 $ x 8 %

1.000 $ x 5 %

1) Writing the program

	Shares	Bonds	Constraints

„Due to the risk of investment, it was decided not to invest more than 30% of the available funds in shares."

Max. invest. in shares	1.500	0	**300.000**

The measuring unit must be the same!
$ → $
pieces / number of... → $

1.000.000 $ x 30 %

1) Writing the program

	Shares	**Bonds**	Constraints

„ It is therefore necessary to place at least part of the funds into bonds, but taking into account that each individual investor can not buy less than 500 bonds."

Min. amount of bonds	0	1	500

The measuring unit must be the same!
amount [pieces] → amount
$ → pieces

1) Writing the program

	Shares	Bonds	Constraints

„They are considering to invest into shares at a price of $ 1,500.00 with an annual yield of 8 % and bonds at a price of $ 1,000.00 with an annual yield of 5 %.”
„An optimal investment strategy (number of shares and bonds) should be determined if the goal is to maximize total annual earnings…”

Godišnji prinos	8%	5%	
Zarada po vrsti v.p.	120	50	

1.500 $ x 8 %

1.000 $ x 5 %

1) Writing the program

	Shares	**Bonds**	Constraints
Budget	1.500,00	1.000,00	1.000.000,00
Profit next year	120,00	50,00	40.000,00
Max. Invest. Shares	1.500,00		300.000,00
Min. Invest. Bonds		1	500
MAX EARNING	120	50	

2) Model

A. Objective function

B. Variables (from the objective function)

C. Constraints

A. Objective function

„An optimal investment strategy (number of shares and bonds) should be determined if the goal is to **maximize total annual earnings**, taking into account all set limits."

	Max. Earnings
Objective	0

=
(sales price of a single stock x amount of stocks)
+
(sales price of a single bond x amount of bonds)

*The value is still 0 because we still don't know the final values of the variables.

B. Variables

- Shares and bonds

	Shares	Bonds
Variables	0	0

The amounts are 0 shares and 0 bonds → Solver will give us the optimal solution amounts!

C. Constraints

	Shares	Bonds	Constraints
Budget	1.500,00	1.000,00	1.000.000,00
Profit next year	120,00	50,00	40.000,00
Max. Invest. Shares	1.500,00		300.000,00
Min. Invest. Bonds		1	500
MAX EARNING	120	50	

Constraints	LS	Ratio	RS

LEFT SIDE
= the values with the optimal solution

≥
=
≤

RIGHT SIDE
= the values of the contraints

C. Constraints

	Shares	Bonds	Constraints
Budget	1.500,00	1.000,00	1.000.000,00
Profit next year	120,00	50,00	40.000,00
Max. Invest. Shares	1.500,00		300.000,00
Min. Invest. Bonds		1	500
MAX EARNING	120	50	

Constraints	LS	ratio	RS
Budget	0	<=	1.000.000,00
Profit next year	0	>=	40.000,00
Max. Invest. Shares	0	<=	300.000,00
Min. Invest. Bonds	0	>=	500

C. Constraints

	Shares	Bonds	Constraints
Budget	1.500,00	1.000,00	1.000.000,00
Profit next year	120,00	50,00	40.000,00
Max. Invest. Shares	1.500,00		300.000,00
Min. Invest. Bonds		1	500
MAX EARNING	120	50	

	Shares	Bonds
Variables	0	0

Constraints	LS	ratio	RS
Budget	0	<=	
Profit next year	0	>=	
Max. Invest. Shares	0	<=	
Min. Invest. Bonds	0	>=	500,00

= (price of a share x amount of shares)
+
(price of a bond x amoutn of bonds)

C. Constraints

	Shares	Bonds	Constraints
Budget	1.500,00	1.000,00	1.000.000,00
Profit next year	120,00	50,00	40.000,00
Max. Invest. Shares	1.500,00		300.000,00
Min. Invest. Bonds		1	500
MAX EARNING	120	50	

	Shares	Bonds
Variables	0	0

Constraints	LS	ratio	RS
Budget	0	<=	
Profit next year	0	>=	
Max. Invest. Shares	0	<	
Min. Invest. Bonds	0	>=	500,00

= (profit from a share x amount of shares)
+
(profit from a bond x amoutn of bonds)

C. Constraints

	Shares	Bonds	Constraints
Budget	1.500,00	1.000,00	1.000.000,00
Profit next year	120,00	50,00	40.000,00
Max. Invest. Shares	1.500,00		300.000,00
Min. Invest. Bonds		1	500
MAX EARNING	120	50	

	Shares	Bonds
Variables	0	0

Constraints	LS	ratio	RS
Budget	0	<=	1.000.000,00
Profit next year	0	>	
Max. Invest. Shares	0	<=	300.000,00
Min. Invest. Bonds	0	>=	500,00

= (price of a share x amount of shares)

C. Constraints

	Shares	Bonds	Constraints
Budget	1.500,00	1.000,00	1.000.000,00
Profit next year	120,00	50,00	40.000,00
Max. Invest. Shares	1.500,00		300.000,00
Min. Invest. Bonds		1	500
MAX EARNING	120	50	

	Shares	Bonds
Variables	0	0

Constraints	LS	ratio	RS
Budget	0	<=	1.000.000,00
Profit next year	0	>=	40.000,00
Max. Invest. Shares	0	<=	
Min. Invest. Bonds	0	>=	500,00

= (a bond x amoutn of bonds)

C. Constraints

	Shares	Bonds	Constraints
Budget	1.500,00	1.000,00	1.000.000,00
Profit next year	120,00	50,00	40.000,00
Max. Invest. Shares	1.500,00		300.000,00
Min. Invest. Bonds		1	500
MAX EARNING	120	50	

The total of the investment fund is 1.000.000 $

To make a profit of at least 40.000 $

Not to invest more than 30 % of the available funds

Can not buy less than 500 bonds

		ratio	RS
Budget	0	<=	1.000.000,00
	0	>=	40.000,00
	0	<=	300.000,00
	0	>=	500

	Shares	Bonds	Constraints
Budget	1.500,00	1.000,00	1.000.000,00
Profit next year	120,00	50,00	40.000,00
Max. Invest. Shares	1.500,00		300.000,00
Min. Invest. Bonds		1	500
MAX EARNING	120	50	

Model:

	Max. Earnings
Objective	0

	Shares	Bonds
Variables	0	0

Constraints	LS	ratio	RS
Budget	0	<=	1.000.000,00
Profit next year	0	>=	40.000,00
Max. Invest. Shares	0	<=	300.000,00
Min. Invest. Bonds	0	>=	500,00

↑ The model is set ↑

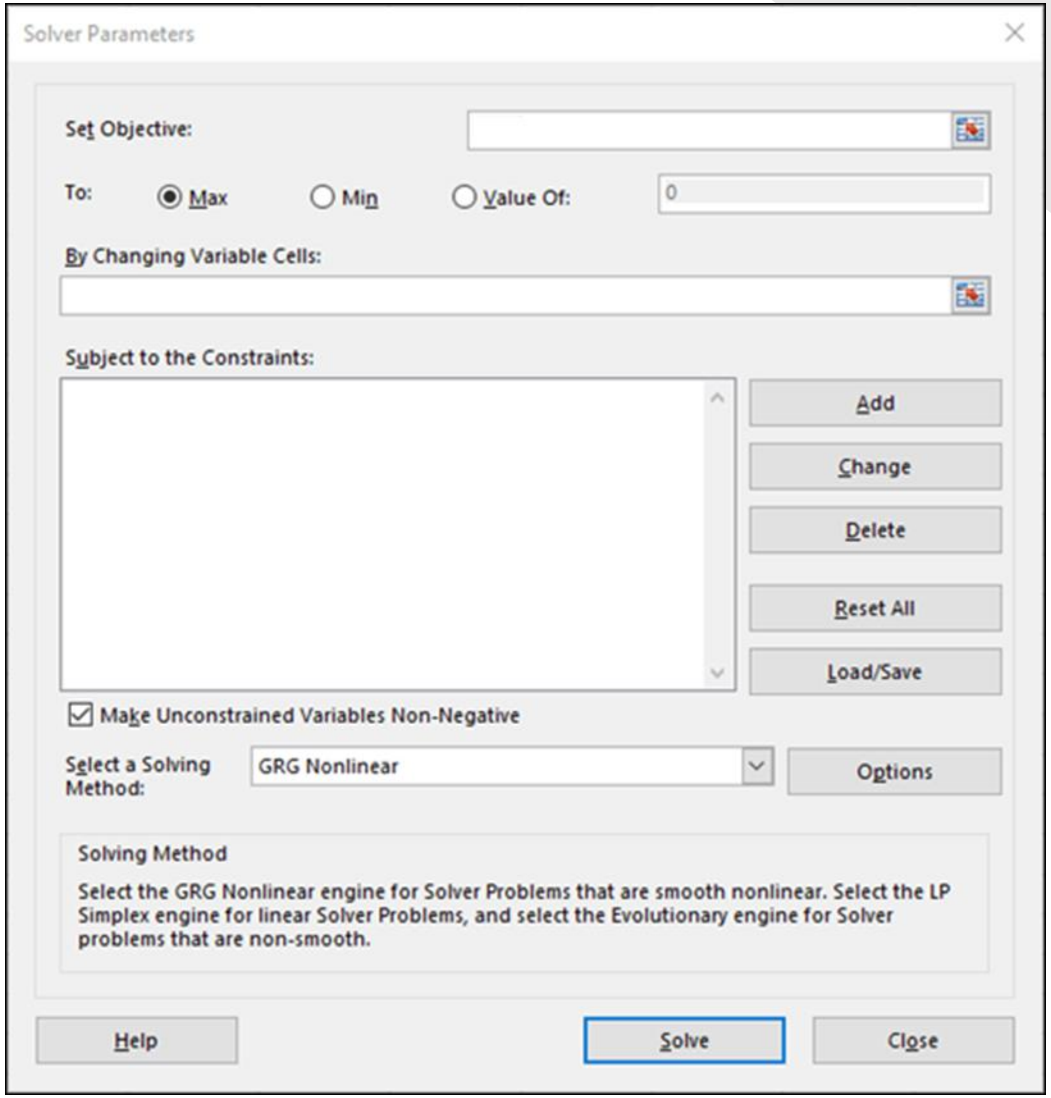
Solver

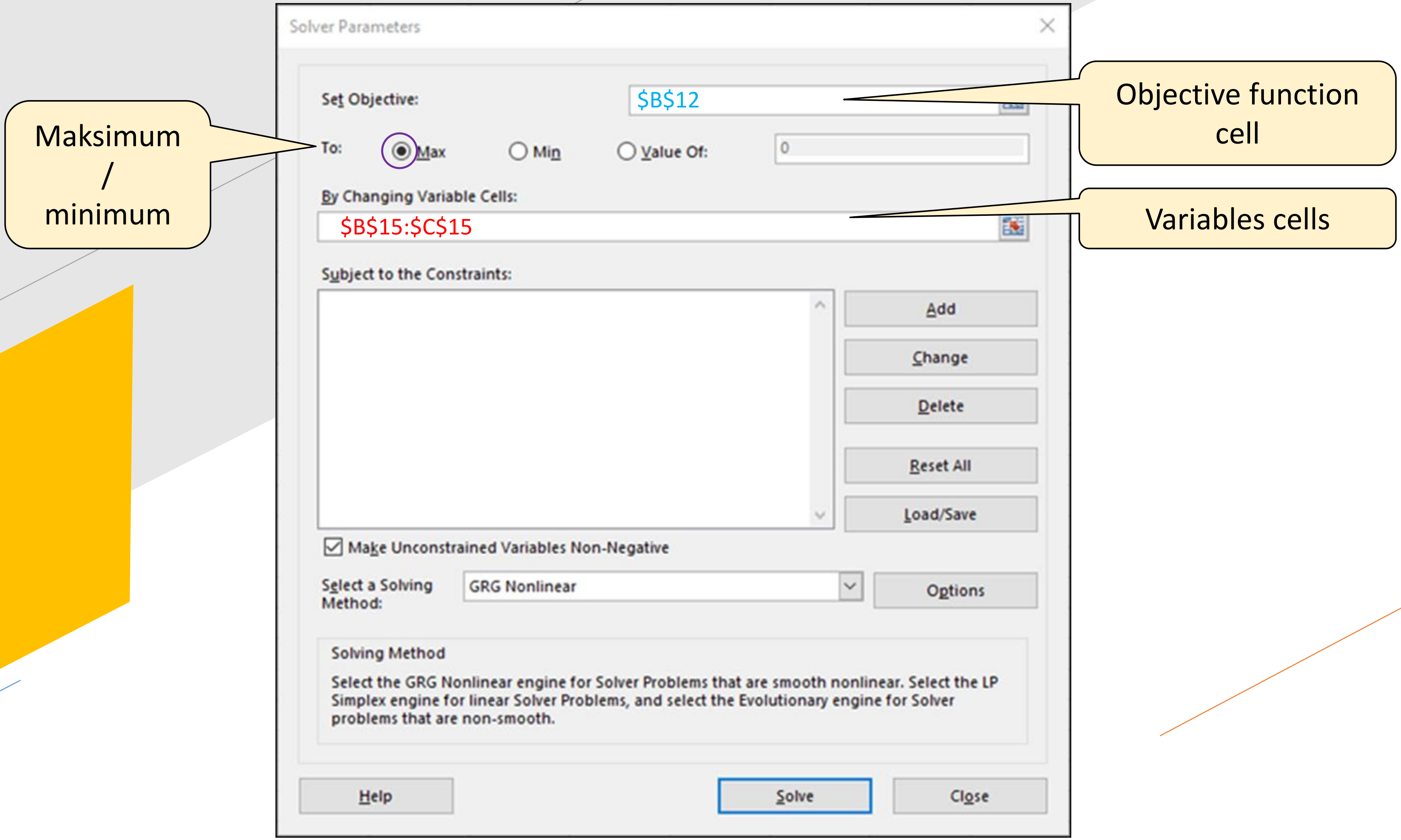
- In the menu bar:

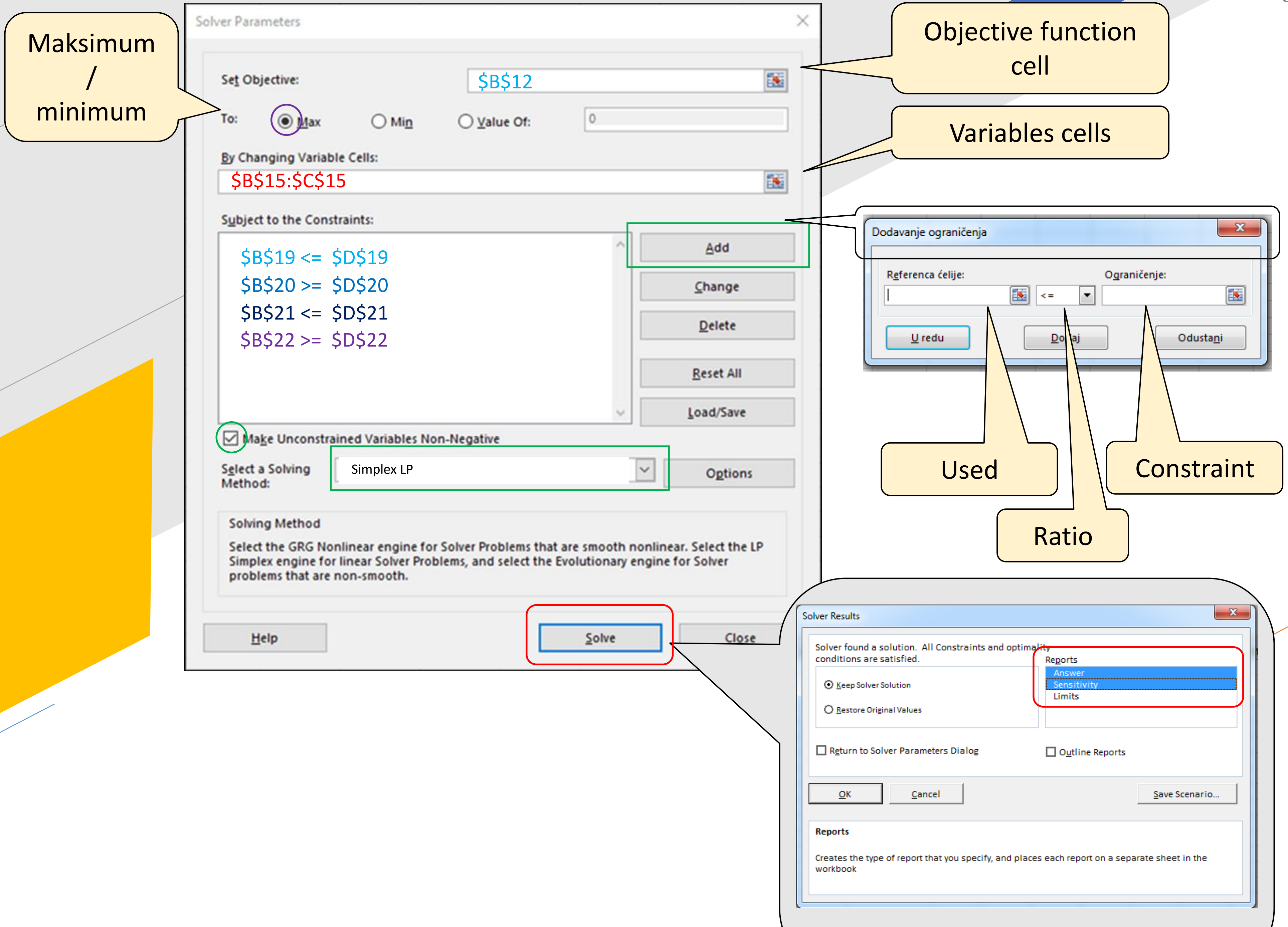
  → Data

  → Solver

  → new pop-up window:




Solver Parameters
Set Objective:
$B$12
To:
Max
Min
Value Of:
0
By Changing Variable Cells:
$B$15:$C$15
Subject to the Constraints:
Add
Change
Delete
Reset All
Load/Save
Make Unconstrained Variables Non-Negative
Select a Solving Method:
GRG Nonlinear
Options
Solving Method
Select the GRG Nonlinear engine for Solver Problems that are smooth nonlinear. Select the LP Simplex engine for linear Solver Problems, and select the Evolutionary engine for Solver problems that are non-smooth.
Help
Solve
Close
Maksimum / minimum
Objective function cell
Variables cells


Maksimum / minimum
Solver Parameters
Set Objective:
$B$12
To:
Max
Min
Value Of:
0
By Changing Variable Cells:
$B$15:$C$15
Subject to the Constraints:
$B$19 <= $D$19
$B$20 >= $D$20
$B$21 <= $D$21
$B$22 >= $D$22
Add
Change
Delete
Reset All
Load/Save
Make Unconstrained Variables Non-Negative
Select a Solving Method:
Simplex LP
Options
Solving Method
Select the GRG Nonlinear engine for Solver Problems that are smooth nonlinear. Select the LP Simplex engine for linear Solver Problems, and select the Evolutionary engine for Solver problems that are non-smooth.
Help
Solve
Close
Objective function cell
Variables cells
Dodavanje ograničenja
Referenca ćelije:
Ograničenje:
<=
U redu
Dodaj
Odustani
Used
Ratio
Constraint
Solver Results
Solver found a solution. All Constraints and optimality conditions are satisfied.
Reports
Answer
Sensitivity
Limits
Keep Solver Solution
Restore Original Values
Return to Solver Parameters Dialog
Outline Reports
OK
Cancel
Save Scenario...
Reports
Creates the type of report that you specify, and places each report on a separate sheet in the workbook

Problem solution

	Shares	Bonds	Constraints
Budget	1.500,00	1.000,00	1.000.000,00
Profit next year	120,00	50,00	40.000,00
Max. Invest. Shares	1.500,00		300.000,00
Min. Invest. Bonds		1	500
MAX EARNING	120	50	

Model:

	Max. Earnings
Objective	59000

	Shares	Bonds
Variables	200	700

Constraints	LS	ratio	RS
Budget	1000000	<=	1.000.000,00
Profit next year	59000	>=	40.000,00
Max. Invest. Shares	300000	<=	300.000,00
Min. Invest. Bonds	700	>=	500,00

What to remember?

- Two-variable linear model solution
- Multi-variable linear model solution

Thank you!

Questions?

Sensitivity analysis

Exercises 4.

International Business

ILO1	convert the practical problem of optimization into a mathematical statement as well as interpret results of applied quantitative models in business decision making

Tools for solving linear models

- more complex multi-variable tasks need to be addressed through available software tools

- Microsoft Excel

→ you need to enable an add-in in the program: Solver

→ *Activation procedure*: File → Options → Add-Ins → Manage: Excel Add-ins → Go… →
✓ Anslysis ToolPak; ✓ Solver Add-in → OK

→ https://support.office.com/en-us/article/load-the-solver-add-in-in-excel-612926fc-d53b-46b4-872c-e24772f078ca

→ https://www.youtube.com/watch?v=W6tIS4JZ5J0

Sensitivity analysis

Sensitivity analysis

→ the information we have in setting up and solving the problem of linear programming can be changed for various reasons

→ changes in input data can significantly affect the changes in the optimum solution

→ **IN PRACTICE**: change of input data is routine → sensitivity analysis equally important as an optimal solution

Sensitivity analysis

1. **Answer report**
   - provides data on the optimal solution, optimal values of variables, and the fulfillment of constraints

2. **Sensitivity report**
   - shows how much input parameters can be changed so that the solution offered remains optimally

EXERCISE 1

Exercise 1.

- The pension fund is considering investing in securities. There is a total investment fund of $ 1.000.000,00 available. They are considering to invest into shares at a price of $ 1.500,00 with an annual yield of 8 % and bonds at a price of $ 1.000,00 with an annual yield of 5 %. Next year it is necessary to make a profit on the securities of at least $ 40.000,00. Due to the risk of investment, it was decided not to invest more than 30 % of the available funds in shares. It is therefore necessary to place at least part of the funds into bonds, but taking into account that each individual investor cannot buy less than 500 bonds.
- An optimal investment strategy (number of shares and bonds) should be determined if the goal is to maximize total annual earnings, taking into account all set limits.

Answer report

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$B$11	Objective Max. Earnings	0	59000

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$14	Variables Shares	0	200	Contin
$C$14	Variables Bonds	0	700	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$B$18	Budget LS	1000000	$B$18<=$D$18	Binding	0
$B$19	Profit next year LS	59000	$B$19>=$D$19	Not Binding	19000
$B$20	Max. Invest. Shares LS	300000	$B$20<=$D$20	Binding	0
$B$21	Min. Invest. Bonds LS	700	$B$21>=$D$21	Not Binding	200

Sensitivity report

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$14	Variables Shares	200	0	120	1E+30	45
$C$14	Variables Bonds	700	0	50	30	50

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$B$18	Budget LS	1000000	0,05	1000000	1E+30	200000
$B$19	Profit next year LS	59000	0	40000	19000	1E+30
$B$20	Max. Invest. Shares LS	300000	0,03	300000	200000	300000
$B$21	Min. Invest. Bonds LS	700	0	500	200	1E+30

OBJECTIVE FUNCTION

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$B$11	Objective Max. Earnings	0	59000

MaxZ

The maximum profit earned is **59.000 USD**.

OPTIMAL SOLUTION

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$14	Variables Shares	0	200	Contin
$C$14	Variables Bonds	0	700	Contin

X1
X2

The optimal solution is the investment into **200 shares** and **700 bonds**.

CONSTRAINTS

X3; X4; X5; X6

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$B$18	Budget LS	1000000	$B$18<=$D$18	Binding	0
$B$19	Profit next year LS	59000	$B$19>=$D$19	Not Binding	19000
$B$20	Max. Invest. Shares LS	300000	$B$20<=$D$20	Binding	0
$B$21	Min. Invest. Bonds LS	700	$B$21>=$D$21	Not Binding	200

REAL VALUE

SLACK VALUE

With the optimal investment strategy:

- We used all of the budget we had of 1.000.000 $ (x3)
- We surpassed our goal income of 40.000 $ by additional 19.000 $ (x4)
- We invested the maximal possible amount of 300.000 $ into stocks (x5)
- We bought 200 more bonds than the minimum purchase possibility (x6)

Sensitivity to parameter changes in objective function

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$B\$14	Variables Shares	200	0	120	1E+30	45
\$C\$14	Variables Bonds	700	0	50	30	50

STRUCTURAL VARIABLES OPRIMAL VALUES

OBJECTIVE FUNCTION COEFFICIENTS

ALLOWED CHANGE OF OBJECTIVE FUNCTION COEFFICIENTS
so that the
OPTIMAL SOLUTION IS STILL VALID (it does not change)

Allowable increase / decrease

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$14	Variables Shares	200	0	120	1E+30	45
$C$14	Variables Bonds	700	0	50	30	50

↓X1… 120 - 45 = 75 ↑X1… 120 + ∞ = + ∞

Shares [X1]:

The return on shares investment can increase infinitely (with all other parameters unchanged) while maintaining optimal investment in 200 shares and 700 bonds.

The return on shares investment can be reduced by up to $ 45 (with all other parameters unchanged) while maintaining optimal investment in 200 shares and 700 bonds.

The range within which the return on shares investment can vary (assuming that the rest does not change) so that the optimal investment remains unchanged ranges from $ 75 to $ ∞.

Allowable increase / decrease

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$14	Variables Shares	200	0	120	1E+30	45
$C$14	Variables Bonds	700	0	50	30	50

↓X2… 50 – 50 = 0 ↑X2… 50 + 30 = 80

Bonds [X2]:

The return on bonds investment can be increased by up to $ 30 (with all other parameters unchanged) while maintaining optimal investment in 200 shares and 700 bonds.

The return on bonds investment can be reduced by up to $ 50 (with all other parameters unchanged) while maintaining optimal investment in 200 shares and 700 bonds.

The range within which the return on bonds investment can vary (assuming that the rest does not change) so that the optimal investment remains unchanged ranges from $ 0 to $ 80.

Reduced cost

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$14	Variables Shares	200	0	120	1E+30	45
$C$14	Variables Bonds	700	0	50	30	50

REDUCED COST

→ if we give up investing in one security, what is the cost of the missed opportunity?

In this example, the opportunity cost is zero because the investor has invested in both types of securities (neither structural variable is zero).

Reduced cost *EXAMPLE

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$14	Variables Shares	0	80	120	1E+30	45
$C$14	Variables Bonds	1.000	0	50	30	50

By giving up on share investing, the opportunity cost for the investor is $ 80 per non-purchased share.

Sensitivity to changes in the vector of free members

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$B$18	Budget LS	1000000	0,05	1000000	1E+30	200000
$B$19	Profit next year LS	59000	0	40000	19000	1E+30
$B$20	Max. Invest. Shares LS	300000	0,03	300000	200000	300000
$B$21	Min. Invest. Bonds LS	700	0	500	200	1E+30

SHADOW PRICE

The amount of change in the value of the goal function that would occur if the right side of that limit were increased by 1 (and all other parameters remained the same)

ALLOWABLE INCREASE/ DECREASE OF THE CONSTRAINT RIHGT SIDE

The amount by which we can increase / decrease the right side of the constraint (with the other parameters unchanged) without changing the dual price

Shadow price

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$B$18	Budget LS	1000000	0,05	1000000	1E+30	200000
$B$19	Profit next year LS	59000	0	40000	19000	1E+30
$B$20	Max. Invest. Shares LS	300000	0,03	300000	200000	300000
$B$21	Min. Invest. Bonds LS	700	0	500	200	1E+30

Any additional $ 1 invested in securities would give the investor a $ 0.05 return.

Increasing the minimum annual earnings limit by $ 1 will not affect the total return on investment.

Any additional $ 1 invested in shares would give the investor a return of $ 0.03.

Increasing the minimum quantity of buyed bonds limit by 1 will not affect the total return on investment.

Shadow price

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$B$18	Budget LS	1000000	0,05	1000000	1E+30	200000
$B$19	Profit next year LS	59000	0	40000	19000	1E+30
$B$20	Max. Invest. Shares LS	300000	0,03	300000	200000	300000
$B$21	Min. Invest. Bonds LS	700	0	500	200	1E+30

The available budget can be increased infinitely so that the dual price remains $ 0.05.

The available budget can be reduced by up to $ 200,000 so that the dual price remains $ 0.05.

The available budget can range from $ 800,000 to infinitely many $ to keep the dual price $ 0.05.

Shadow price

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$B$18	Budget LS	1000000	0,05	1000000	1E+30	200000
$B$19	Profit next year LS	59000	0	40000	19000	1E+30
$B$20	Max. Invest. Shares LS	300000	0,03	300000	200000	300000
$B$21	Min. Invest. Bonds LS	700	0	500	200	1E+30

The minimum annual earnings limit can be increased by up to $ 19.000 so that the dual price remains $ 0.

The minimum annual earnings limit can be reduced infinitely so that the dual price remains $ 0.

The minimum annual earnings limit can range from $ 0 to $ 59.000 to keep the dual price $ 0.

Shadow price

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$B$18	Budget LS	1000000	0,05	1000000	1E+30	200000
$B$19	Profit next year LS	59000	0	40000	19000	1E+30
$B$20	Max. Invest. Shares LS	300000	0,03	300000	200000	300000
$B$21	Min. Invest. Bonds LS	700	0	500	200	1E+30

The available budget for shares investment can be increased by up to $ 200.000 so that the dual price remains $ 0.03.

The available budget can be reduced by up to $ 300.000 so that the dual price remains $ 0.03.

The available budget can range from $ 0 to $ 500.000 to keep the dual price $ 0.03.

Shadow price

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$B$18	Budget LS	1000000	0,05	1000000	1E+30	200000
$B$19	Profit next year LS	59000	0	40000	19000	1E+30
$B$20	Max. Invest. Shares LS	300000	0,03	300000	200000	300000
$B$21	Min. Invest. Bonds LS	700	0	500	200	1E+30

Increasing the minimum quantity of buyed bonds limit by 1 will not affect the total return on investment.

The minimum quantity of bonds can be increased by 200 so that the dual price remains $ 0.

The minimum quantity of bonds can be reduced infinitely so that the dual price remains $ 0.

The minimum quantity of bonds can range from 0 to 700 bonds to keep the dual price $ 0.

What to remember?

- Sensitivity analysis
- Answer report, sensitivity repost
- Reduced costs, shadow prices

Anderson: Quantitative methods for business decisions

Thank you!

Questions?

Preparation for the first preliminary exam

Exercises 5.

International Business

ILO1	convert the practical problem of optimization into a mathematical statement as well as interpret results of applied quantitative models in business decision making

Preliminary exam 1:

Exercise tasks:

1.	**Solving a linear programming problem using the graphical solution method**	
	• Formulation of the given problem (general and canonical form); interpretation of structural and slack variables • Graphical solution of the problem; interpretation of calculated results	**12,5 points**
2.	**Solving a linear programming problem using Solver**	
	• Formulation and solution of the given problem in Excel • Answer report and sensitivity analysis	**12,5 points**

Exercise 1

- The small factory produces two types of screws V1 and V2. For 1 kg of V1 it is necessary to work on machine S1 for 2 h, and for 1 kg of V2 it is necessary to work on machine S1 for 1 h, and on machine S2 for 4 h. The capacities of the machines are limited: machine S1 10 h, and machine S2 12 h. What quantity of screws needs to be produced in order to maximize the profit, if HRK 20 is obtained for a kilogram of V1 and HRK 30 for a V2, provided that at least 2 kg of V1 is placed on the market?
- Mathematically formulate the problem of linear programming by setting the objective function, taking into account the set limits.
- Write the canonical form of the model and interpret the meaning of the structural and slack variables.
- Graphically solve the problem and find the optimal solution.
- Interpret the solution obtained by answering the following questions:
  - What are the optimal quantities of screws produced?
  - How much income was generated?
  - What is the situation with the model limitations?

Exercise 1

- Mathematically formulate the problem of linear programming by setting the objective function, taking into account the set limits.

	V1	V2	Constraints:
Machine S1	2	1	10 h
Machine S2	0	4	12 h
Min. Q of V1	1	0	2 kg
Price	20 kn	30 kn	← MAX!

GENERAL FORM:

$$MaxZ = 20x_1 + 30x_2$$

$$2x_1 + x_2 \leq 10$$
$$4x_2 \leq 12$$
$$x_1 \geq 2$$

$$x_1, x_2 \geq 0$$

Exercise 1

- Write the canonical form of the model and interpret the meaning of the structural and slack variables.

CANONICAL FORM:

$$MaxZ = 20x_1 + 30x_2 + 0x_3 + 0x_4 + 0x_5$$

$$2x_1 + x_2 + x_3 = 10$$
$$4x_2 + x_4 = 12$$
$$x_2 - x_5 = 2$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Interpretation of variables:

STRUCTURAL:

$\boldsymbol{x_1}$: ammount of screws V1 in kg

$\boldsymbol{x_2}$: ammount of screws V2 in kg

SLACK:

$\boldsymbol{x_3}$: unused working hours of machine S1

$\boldsymbol{x_4}$: unused working hours of machine S2

$\boldsymbol{x_5}$: overslow over min. required ammount of V1 screws in kg

Exercise 1

- Graphically solve the problem and find the optimal solution.

1st constraint:

p1: $2x_1 + x_2 \leq 10$

$x_1 = 0$

$x_2 = 10$

[0; 10]

$x_2 = 0$

$2x_1 = 10 \quad /:2$

$x_1 = 5$

[5; 0]

2nd constraint:

p2: $4x_2 \leq 12$

$x_2 = 3$

[0; 3]

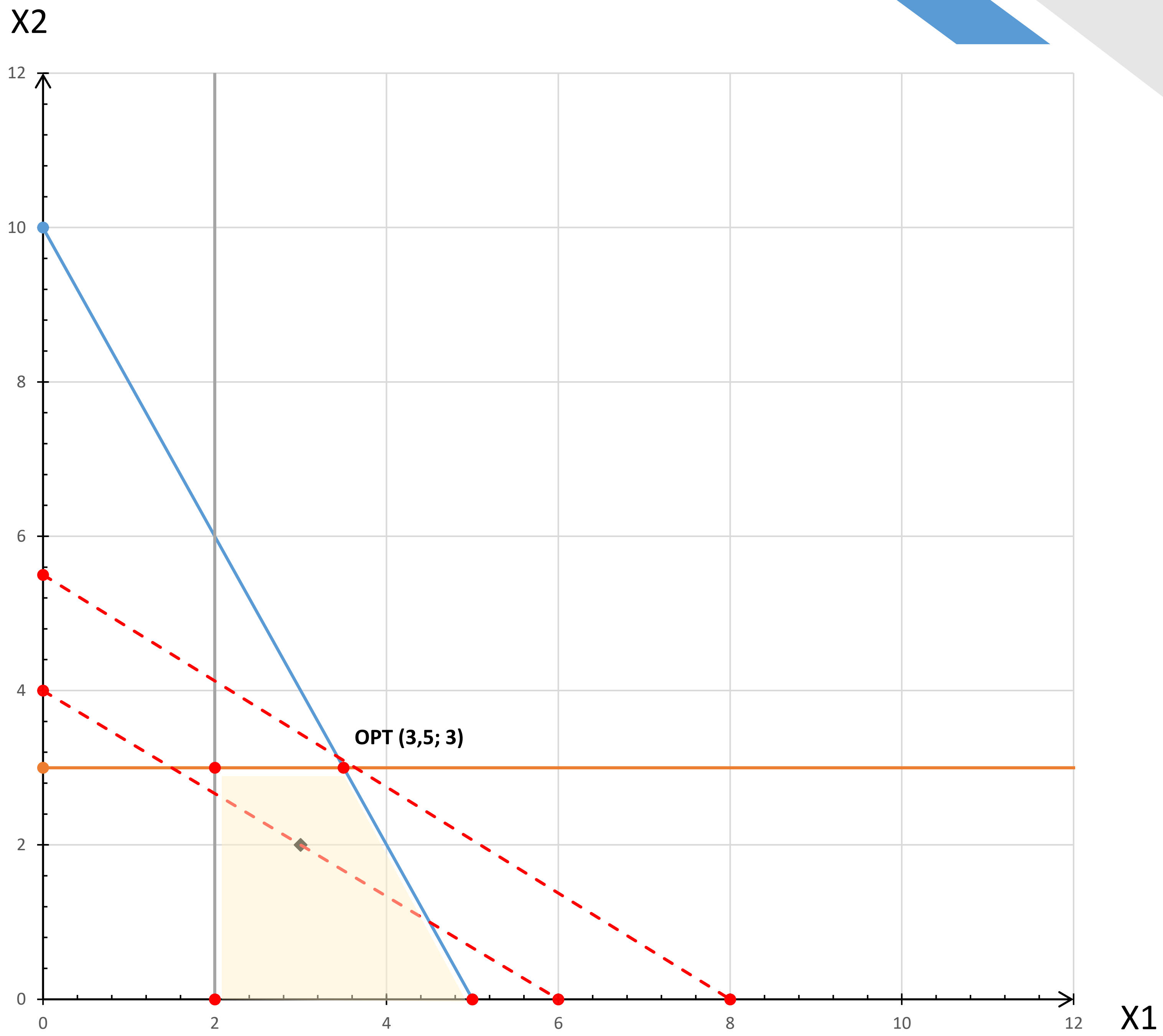
3rd constraint:

p3: $x_1 \geq 2$

$x_1 = 2$

[2; 0]

Exercise 1



T [3; 2]

$$MaxZ = 20x_1 + 30x_2$$

$$20 \times 3 + 30 \times 2 = 120$$

$$20x_1 + 30x_2 = 120$$

$x_1 = 0, \quad x_2 = 4$ [0; 4]

$x_2 = 0, \quad x_1 = 6$ [6; 0]

Exercise 1

- Interpret the solution obtained by answering the following questions:
  - What are the optimal quantities of screws produced?

    OPT [3,5; 3]

    The optimal amounts of production are 3,5 kg of V1 screws and 3 kg of V2 screws.
  - How much income was generated?

    $$MaxZ = 20 \times 3{,}5 + 30 \times 3 = 160$$

    The maximum profit is 160 HRK.
  - What is the situation with the model limitations?

    $$2 \times 3{,}5 + 3 = 10 = 10$$

    $$4 \times 3 = 12 = 12$$

    $$3{,}5 > 2$$

    The available working hours of both machines are used up entirely.
    The minimum requirement of screws V1 production is surpassed by 1,5 kg.

Exercise 2

- A professional athlete wants to put together a diet plan that would meet his daily needs for nutrients: carbohydrates, fats and proteins. He decided to ingest the same through two types of meals a day. The first type of meal contains 40 g of carbohydrates, 20 g of protein and 8 g of fat, while the second type of meal contains 30 g of carbohydrates, 40 g of protein and 8 g of fat. The price of the first meal is 10 HRK, and the second 12 HRK. The trainer prescribed him that he must consume a minimum of 250 g of carbohydrates, 250 g of protein and 55 g of fat per day. Help him determine the amounts of both types of meals so that he meets the minimum required amounts of nutrients, while minimizing his cost.
- Mathematically formulate the problem of linear programming by setting the objective function, taking into account the set limits.
- Write the canonical form of the model and interpret the meaning of the structural and slack variables.
- Graphically solve the problem and find the optimal solution.
- Interpret the solution obtained by answering the following questions:
  - What are the optimal quantities of meals?
  - How much are the minimal costs?
  - What is the situation with the model limitations?

Exercise 2

- Mathematically formulate the problem of linear programming by setting the objective function, taking into account the set limits.

	Meal 1	Meal 2	Constraints:
Carbonhydrates (g)	40	30	250
Protein (g)	20	40	250
Fat (g)	8	8	55
Price (HRK)	10	12	← MIN!

GENERAL FORM:

$$MinW = 10x_1 + 12x_2$$

$$40x_1 + 30x_2 \geq 250$$
$$20x_1 + 40x_2 \geq 250$$
$$8x_1 + 8x_2 \geq 55$$

$$x_1, x_2 \geq 0$$

Exercise 2

- Write the general form of the model and interpret the meaning of the structural and slack variables.

CANONICAL FORM:

$$MinW = 10x_1 + 12x_2 + 0x_3 + 0x_4 + 0x_5$$

$$40x_1 + 30x_2 - x_3 = 250$$
$$20x_1 + 40x_2 - x_4 = 250$$
$$8x_1 + 8x_2 - x_5 = 55$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Interpretation of variables:

STRUCTURAL:

$\boldsymbol{x_1}$: ammount of meal type 1
$\boldsymbol{x_2}$: ammount of meal type 2

SLACK:

$\boldsymbol{x_3}$: overflow over min. required ammount of carbonhydrates in g
$\boldsymbol{x_4}$: overflow over min. required ammount of protein in g
$\boldsymbol{x_5}$: overflow over min. required ammount of fat in g

Exercise 2

- Graphically solve the problem and find the optimal solution.

1st constraint:

p1: $40x_1 + 30x_2 \geq 250$

$x_1 = 0$
$x_2 = 8{,}33$

$x_2 = 0$
$x_1 = 6{,}25$

2nd constraint:

p2: $20x_1 + 40x_2 \geq 250$

$x_1 = 0$
$x_2 = 6{,}25$

$x_2 = 0$
$x_1 = 12{,}5$

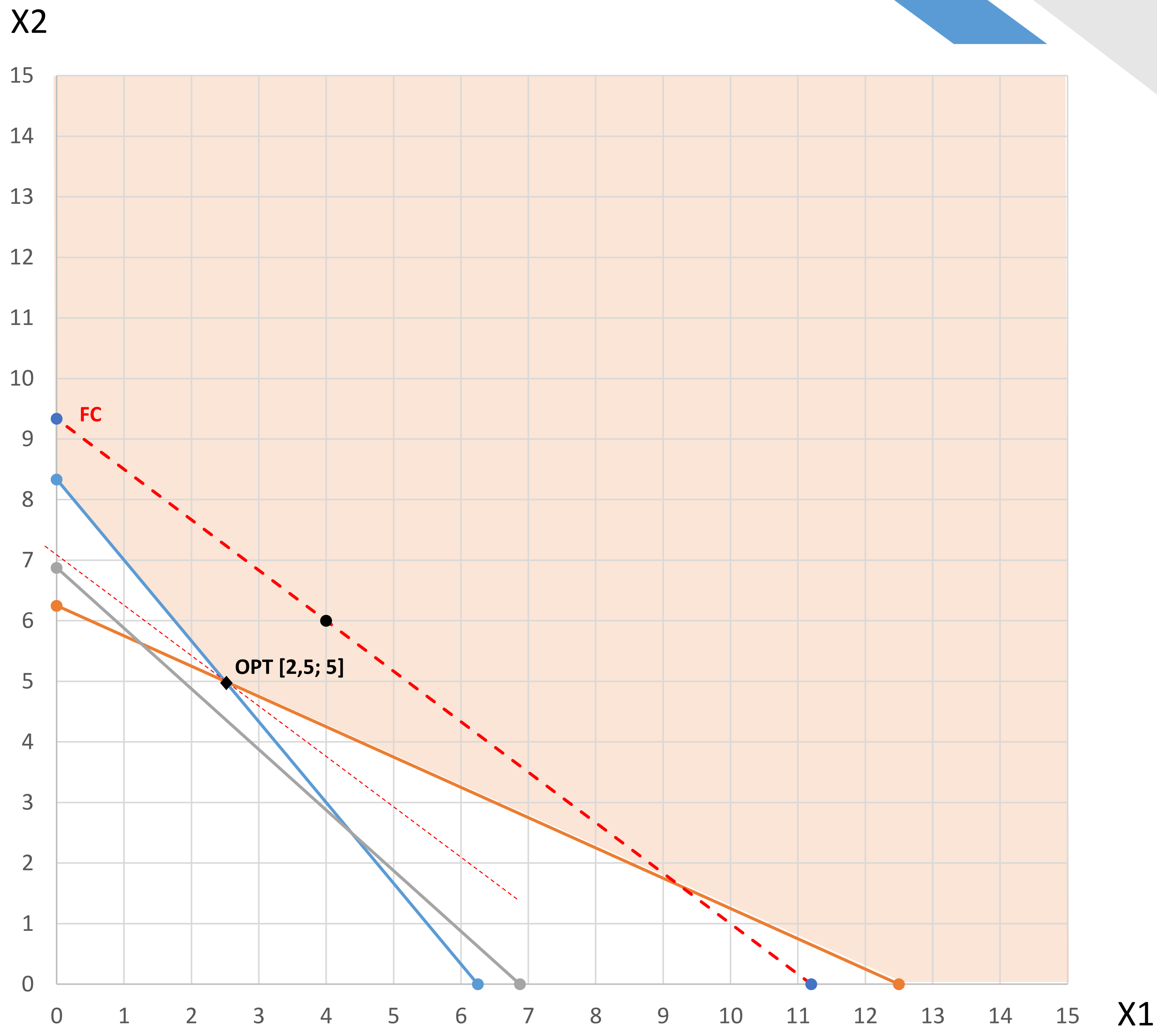
3rd constraint:

p3: $8x_1 + 8x_2 \geq 55$

$x_1 = 0$
$x_2 = 6{,}88$

$x_2 = 0$
$x_1 = 6{,}88$

Exercise 2



T [4; 6]

$$MinW = 10x_1 + 12x_2$$

$$10 \times 4 + 12 \times 6 = 112$$

$$10x_1 + 12x_2 = 112$$

$x_1 = 0, \ \ x_2 = 9{,}33$ [0; 9,33]

$x_2 = 0, \ \ x_1 = 11{,}2$ [11,2; 0]

Exercise 2

- Interpret the solution obtained by answering the following questions:
  - What are the optimal quantities of meals?

    OPT [2,5; 5]

    The optimal amounts of food are 2,5 meals type 1 and 5 meals type 2.
  - How much are the minimal costs?

    $MinW = 10x_1 + 12x_2 = 10 \times 2{,}5 + 12 \times 5 = 85$

    The minimum cost is 85 HRK.
  - What is the situation with the model limitations?

    $40x_1 + 30x_2 \geq 250 \rightarrow 40 \times 2{,}5 + 30 \times 5 = 250$

    $20x_1 + 40x_2 \geq 250 \rightarrow 20 \times 2{,}5 + 40 \times 5 = 250$

    $8x_1 + 8x_2 \geq 55 \rightarrow 8 \times 2{,}5 + 8 \times 5 = 60$

The minimum requirement of fat is surpassed by 5 g, while the ammount of carbonhydrates and protein did not surpass the minimally required ammounts.

EXERCISE 3

Exercise 3

- The company is planning an advertising campaign to attract new customers and wants to place a total of no more than 10 ads in three daily newspapers. Each ad in newspaper A costs $ 200 and will be read by 2,000 people. Each ad in newspaper B costs $ 100 and will be read by 500 people. Each newspaper C ad costs $ 100 and will be read by 1,500 people. The company wants the ads to be read by at least 16,000 people in total. Determine the number of ads in each newspaper which the company will place in order to minimize advertising costs, if it is a known fact that newspaper C cannot publish more than 4 advertisements.
- Formulate the linear programming problem mathematically. Solve the problem using Excel. (Exercises 5 – Solutions.xlsx)

Exercise 3

- Based on the answer report and the sensitivity report, answer the following questions:
  - How much does it cost to advertise the company?
  - In which newspapers did the company decide to place its advertisements and how many?
  - Are all restrictions met? Is there an overflow or unused resources in the limitations?
  - Are there opportunity costs? If so, what are they saying?
  - What is the price level of the advertisement in newspaper A, so that the company still decides to place 5 advertisements in that newspaper and 4 advertisements in newspaper C?
  - How will an allowable increase in the number of ads affect the overall cost of advertising?
  - If the desired minimum number of people who see the ads increases by a 1,000, what impact will this have on the total cost of advertising?
  - If newspaper C allowed more advertisements, how would that affect the company’s costs?

Exercise 3

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$B$9	OF Min. Costs	0	1400

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$12	Variables Newspaper A	0	5	Contin
$C$12	Variables Newspaper B	0	0	Contin
$D$12	Variables Newspaper C	0	4	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$B$16	Max. Advertisements LS	9	$B$16<=$D$16	Not Binding	1
$B$17	Min. Readers LS	16000	$B$17>=$D$17	Binding	0
$B$18	Max. Ads in n. C LS	4	$B$18<=$D$18	Binding	0

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$12	Variables Newspaper A	5	0	200	200	66,66666667
$C$12	Variables Newspaper B	0	50	100	1E+30	50
$D$12	Variables Newspaper C	4	0	100	50	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$B$16	Max. Advertisements LS	9	0	10	1E+30	1
$B$17	Min. Readers LS	16000	0,1	16000	2000	10000
$B$18	Max. Ads in n. C LS	4	-50	4	4	4

Exercise 3

- Based on the answer report and the sensitivity report, answer the following questions:
  - How much does it cost to advertise the company?

  The minimum cost is $ 1,400.

  - In which newspapers did the company decide to place its advertisements and how many?

  The company paid for the publication of 5 advertisements in newspaper A and 4 advertisements in list B.

  - Are all restrictions met? Is there an overflow or unused resources in the limitations?

  A total of 9 advertisements were paid, 1 less than the maximum possible number.

  - Are there opportunity costs? If so, what are they saying?

  Yes, we did not decide to place advertisements in newspaper B. The opportunity cost is $ 150.

Exercise 3

- Based on the answer report and the sensitivity report, answer the following questions:
  - What is the price level of the advertisement in newspaper A, so that the company still decides to place 5 advertisements in that newspaper and 4 advertisements in newspaper C?

  The price of an advertisement can range from \$ 143.33 to \$ 400 (200 - 66.67 = \$ 143.33; 200 + 200 = \$ 400).

  - How will an allowable increase in the number of ads affect the overall cost of advertising?

  An increase in the possible number of ads will not affect the amount of the minimum cost (dual price: 0; allowable increase: infinite). With the current budget, it is possible to pay for 9 advertisements.

  - If the desired minimum number of people who will see an ad increases by 1,000, what impact will this have on the total cost of advertising?

  The minimum advertising cost will increase by \$ 10 (1,000 * 0.1 = \$ 10).

  - If newspaper C allowed more advertisements, how would that affect the company's costs?

  Up to the number of 8 advertisements in newspaper C with each additionally published advertisement in that list, the total costs would be reduced by \$ 50.

EXERCISE 4

Exercise 4

The company produces four products: A, B, C and D. In the final part of the manufacturing process, assembly, packaging and stacking operations are performed. The time required to perform each of these operations, in minutes, is shown in the table. The same table shows the profit per piece of each of the products.

Product	**A**	**B**	**C**	**D**
Assembly	2	4	3	7
Packaging	3	2	3	4
Stacking	2	3	2	5
Profit($)	15	25	30	45

The company has 100,000 minutes per year for assembly, 50,000 minutes for packaging and 60,000 minutes for stacking. Determine the annual production plan of individual products for which the company will make the most profit.

a) Formulate and solve the given problem using MS Excel Solver.

Exercise 4

b) Based on the answer report and the sensitivity report, answer the following questions:

- What is the maximum annual profit and how will it be realized?

The maximum profit is 580.000 $ and it will be achieved through the production of 16.000 products B and 6.000 products C.

- Are there any unused capacities? If so, what and how much?

There are 18.000 unused minutes for the assembly process.

- Explain the meaning of reduced cost for product A? How does it affect overall profits?

For each product A they did not produced they gave up on 15 $ of profit.

- What is the range of profit per product sold C, so that it is still worthwhile for the company to sell the currently optimal quantities of the product?

The range of profit is between 25 $ and 37,5 $ for each product C, so that the production will still stay the same (0 A products, 16.000 B products, 6.000 C products and 0 D products).

- What is the meaning of the shadow price for the product packaging constraint?

Each aidtional minute of packiging available would rise the profits by 8$.

Thank you!

Questions?

Regression analysis: assumptions of classical model; OLS, PRF, SRF

Exercises 6.

International Business

ILO2	test the statistical significance of the regression parameters and the applied regression model and to economically interpret the results obtained with the help of available program tools

What is econometrics?

- *“Econometrics is too mathematical; it’s the reason my best friend isn’t majoring in economics.”*
- *“There are two things you are better off not watching in the making: sausages and econometric estimates.”*
- *“Econometrics may be defined as the quantitative analysis of actual economic phenomena.”*

- <u>Econometrics</u> — literally: “economic measurement” — is the quantitative measurement and analysis of actual economic and business phenomena. It attempts to quantify economic reality and bridge the gap between the abstract world of economic theory and the real world of human activity.

Regression analysis

- Simple linear regression model
- Regressand, regressor, disturbance, random sampling
- Population regression function (PRF), sample regression function (SRF)
- Ordinary least squares estimator (OLS), fitted values and residuals

Regression analysis

- **Regression analysis** = a statistical technique that attempts to “explain” movements in one variable – the **dependent variable**, as a function of movements in a set of other variables – the **independent** (or explanatory) **variables**

Regression model

- Simple regression; multiple regression

$Y_i = \beta_0 + \beta_1$; $Y_i = \beta_0 + \beta_1 - \beta_2 + \beta_3$

Dependent variable

Constant / intercept

Constant / intercept

Coefficient of explanatory variable / slope

- Statistical data:
  - Time series data→ ***t***
  - Cross sectional data→ ***i***

Regression model

- Sample regression function (SRF)

$$Y_i = \widehat{\beta_0} + \widehat{\beta_1} x_i + e_i$$

Disturbance / error term

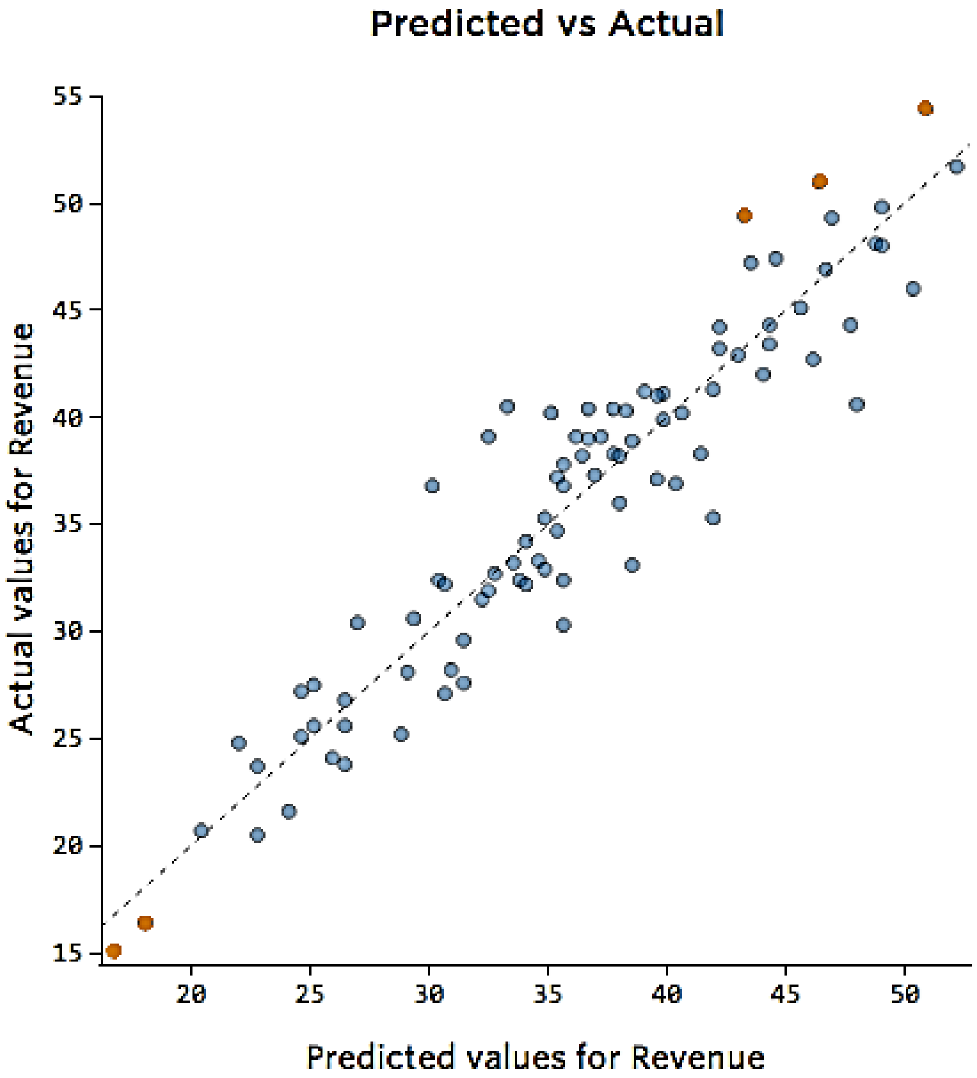
- Population regression function (PRF)

$$Y_i = \beta_0 + \beta_1 x_i + u_i$$

Disturbance / error term

Regression model

- Disturbance:



EXERCISE 1

EXERCISE 1.

- A model is determined:

$$\hat{Y}_t = 18.846{,}4 - 247{,}92X_t$$

- Where:

Y is the number of products sold

X the price of the product

- The data was collected for a period of 12 months.

EXERCISE 1.

a) Explain the economic significance of the assessed parameters. Find out if the parameter with the variable X is the one you expected.

b) Why is it written $\hat{Y}_t$ on the left side of the equation, not Yt?

c) If we add the residual to the equation, would we then write on the left?

d) What kind of data, on which model is evaluated, is used here?

e) Is the rated SRF or PRF rated? Explain the answer!

EXERCISE 1.

a) An increase in the price of a product will lead to a decrease in the number of products sold. The economic criterion is met and the sign is in line with expectations because if the price goes up, the number of products sold will decrease.

b) Because there is no residual (e_t) on the right. → the difference between actual and estimated value

c) No, then we would write Yt.

d) Time series (t). → *The cross sectional dana is labeled „i”

e) Sample regression function because 1 product was taken.

*SRF: $Y_i = \widehat{\beta_0} + \widehat{\beta_1} x_i + e_i$

PRF: $Y_i = \beta_0 + \beta_1 x_i + u_i$

EXERCISE 2

EXERCISE 2.

- Data is given:

Y_t	55	70	90	100	90	105	80	110	125	115	130	130
X_t	10	9	8	7	7	7	7	6,5	6	6	5,5	5

- Where Xt is the price of pounds of orange on a given day, and Yt the quantity of sold orange (in kg) the same day in one store.

a) Estimate a linear model using the least squares method.

b) Suppose that the known real values of the parameter $ß_0$ = 210, $ß_1$ = -15. Calculate the residual value and the value of random deviations for each of the twelve observations.

EXERCISE 2.

a) Estimate a linear model using the least squares method.

$$\widehat{\beta_0} \times n + \widehat{\beta_1} \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} Y_i$$

$$\widehat{\beta_0} \sum_{i=1}^{n} X_i + \widehat{\beta_1} \sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n} X_i Y_1$$

	Yt	Xt	Xt^2	Xt*Yt
n = 12	55	10	100	550
	70	9	81	630
	90	8	64	720
	100	7	49	700
	90	7	49	630
	105	7	49	735
	80	7	49	560
	110	6,5	42,25	715
	125	6	36	750
	115	6	36	690
	130	5,5	30,25	715
	130	5	25	650
Σ	**1200**	**84**	**610,5**	**8045**

$$12\,\widehat{\beta_0} + 84\,\widehat{\beta_1} = 1.200 \quad / \times (-7)$$

$$84\widehat{\beta_0} + 610{,}5\,\widehat{\beta_1} = 8.045$$

$$-84\,\widehat{\beta_0} - 588\,\widehat{\beta_1} = -8400$$

$$84\widehat{\beta_0} + 610{,}5\,\widehat{\beta_1} = 8.045$$

$$22{,}5\,\widehat{\beta_1} = -355 \quad / : 22{,}5$$

$$\widehat{\beta_1} = -15{,}7778$$

$$12\,\widehat{\beta_0} + 84 \times (-15{,}7778) = 1.200$$

$$\widehat{\beta_0} = 210.4444$$

Zadatak 2.

a) Estimate a linear model using the least squares method.

$$\begin{bmatrix} \mathrm{n} & X_{\mathrm{i}} \\ X_{\mathrm{i}} & X_{\mathrm{i}}^2 \end{bmatrix} \times \begin{bmatrix} \widehat{\beta_0} \\ \widehat{\beta_1} \end{bmatrix} = \begin{bmatrix} Y_{\mathrm{i}} \\ X_{\mathrm{i}} Y_{\mathrm{i}} \end{bmatrix}$$

$$\begin{bmatrix} \widehat{\beta_0} \\ \widehat{\beta_1} \end{bmatrix} = \begin{bmatrix} \mathrm{n} & X_{\mathrm{i}} \\ X_{\mathrm{i}} & X_{\mathrm{i}}^2 \end{bmatrix}^{-1} \times \begin{bmatrix} Y_{\mathrm{i}} \\ X_{\mathrm{i}} Y_{\mathrm{i}} \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \times A^{*}$$

	Yt	Xt	Xt^2	Xt*Yt
n = 12	55	10	100	550
	70	9	81	630
	90	8	64	720
	100	7	49	700
	90	7	49	630
	105	7	49	735
	80	7	49	560
	110	6,5	42,25	715
	125	6	36	750
	115	6	36	690
	130	5,5	30,25	715
	130	5	25	650
Σ	1200	84	610,5	8045

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \begin{bmatrix}n & X_i\\ X_i & X_i^2\end{bmatrix}^{-1} \times \begin{bmatrix}Y_i\\ X_iY_i\end{bmatrix}$$

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \begin{bmatrix}12 & 84\\ 84 & 610{,}5\end{bmatrix}^{-1} \times \begin{bmatrix}1200\\ 8045\end{bmatrix}$$

$$\begin{bmatrix}12 & 84\\ 84 & 610{,}5\end{bmatrix}^{-1} = \frac{1}{|detA|} \times A^*$$

$$= \frac{1}{|12 \times 610{,}5 - 84 \times 84|} \times \begin{bmatrix}610{,}5 & -84\\ -84 & 12\end{bmatrix}$$

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \frac{1}{|270|} \times \begin{bmatrix}610{,}5 & -84\\ -84 & 12\end{bmatrix} \times \begin{bmatrix}1200\\ 8045\end{bmatrix}$$

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \frac{1}{|270|} \times \begin{bmatrix}610{,}5 \times 1200 - 84 \times 8045\\ -84 \times 1200 + 12 \times 8045\end{bmatrix}$$

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \frac{1}{|270|} \times \begin{bmatrix}56.820\\ -4260\end{bmatrix} \qquad \begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \begin{bmatrix}210{,}4444\\ -15{,}7778\end{bmatrix}$$

EXERCISE 2.

b) Suppose that the known real values of the parameter $ß_0 = 210$, $ß_1 = -15$. Calculate the residual value and the value of random deviations for each of the twelve observations.

$$e = Y_i - \widehat{Y}_i$$

^Yt	Xt	Yt	e
55	10	60	5
70	9	75	5
90	8	90	0
100	7	105	5
90	7	105	15
105	7	105	0
80	7	105	25
110	6,5	112,5	2,5
125	6	120	-5
115	6	120	5
130	5,5	127,5	-2,5
130	5	135	5
1200	**84**	**1260**	**60**

EXERCISE 3

EXERCISE 3.

- Data on Gross Domestic Product Per Capita (GDP) in US $ and % of employed labour force in agriculture in 10 countries are as follows:

Country	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**
GDP $_{pc}$	5	7	7	8	8	12	10	9	8	9
% of employed labour force in agriculture	8	9	9	8	10	3	5	5	6	6

EXERCISE 3.

- Calculate the parameters of the linear function in which you will evaluate the link between % of employees in agriculture (dependency variable Z) and the level of GDP pc (independent variable G).

	Zemlja	Zi	Gi	Gi^2	Gi*Zi
n = 10	A	8	5	25	40
	B	9	7	49	63
	C	9	7	49	63
	D	8	8	64	64
	E	10	8	64	80
	F	3	12	144	36
	G	5	10	100	50
	H	5	9	81	45
	I	6	8	64	48
	J	6	9	81	54
	∑	**69**	**83**	**721**	**543**

$$10\,\widehat{\beta_0} + 83\widehat{\beta_1} = 69 \quad / \times (-8{,}3)$$

$$83\widehat{\beta_0} + 721\,\widehat{\beta_1} = 543$$

$$-83\,\widehat{\beta_0} - 688{,}9\,\widehat{\beta_1} = -572{,}7$$

$$83\widehat{\beta_0} + 721\,\widehat{\beta_1} = 543$$

$$32{,}1\widehat{\beta_1} = -29{,}7 \quad / : 32{,}1$$

$$\widehat{\beta_1} = -0{,}9252$$

$$10\,\widehat{\beta_0} + 83 \times (-0{,}9252) = 69$$

$$\widehat{\beta_0} = 14{,}5794$$

EXERCISE 3.

- Calculate the parameters of the linear function in which you will evaluate the link between % of employees in agriculture (dependency variable Z) and the level of GDP pc (independent variable G).

n = 10	Zemlja	Zi	Gi	Gi^2	Gi*Zi
	A	8	5	25	40
	B	9	7	49	63
	C	9	7	49	63
	D	8	8	64	64
	E	10	8	64	80
	F	3	12	144	36
	G	5	10	100	50
	H	5	9	81	45
	I	6	8	64	48
	J	6	9	81	54
	∑	**69**	**83**	**721**	**543**

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \begin{bmatrix}10 & 83\\ 83 & 721\end{bmatrix}^{-1} \times \begin{bmatrix}69\\ 543\end{bmatrix}$$

$$\begin{bmatrix}10 & 83\\ 83 & 721\end{bmatrix}^{-1} = \frac{1}{|detA|} \times A^{*} = \frac{1}{|10 \times 721 - 83 \times 83|} \times \begin{bmatrix}721 & -83\\ -83 & 10\end{bmatrix}$$

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \frac{1}{|321|} \times \begin{bmatrix}721 & -83\\ -83 & 10\end{bmatrix} \times \begin{bmatrix}69\\ 543\end{bmatrix}$$

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \frac{1}{|321|} \times \begin{bmatrix}721 \times 69 - 83 \times 543\\ -83 \times 69 + 10 \times 543\end{bmatrix}$$

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \frac{1}{|321|} \times \begin{bmatrix}4.680\\ -297\end{bmatrix} \qquad \begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \begin{bmatrix}14{,}5794\\ -0{,}9252\end{bmatrix}$$

EXERCISE 3.

a) Interpret the meaning of the evaluated parameters economically.

If GDPpc increases by \$ 1,000, the % of agricultural labor force employment will decrease by 0.9252%.

b) If some country's GDP pc is \$ 6.000, what is the expected value of employees in the country's agriculture?

Gi = 6.000 \$ Zi = 14,5794 – 0,9252 x 6

Zi = ? Zi = **9,0282 %**

c) Describe the type of statistical data for model evaluation.

We have cross sectional data (10 countries in 1 year).

EXERCISE 4

EXERCISE 4.

- A company is testing the likelihood of future sales representatives to do well at their job. The manager is interested in the extent to which this test can predict the future success of the job. The relevant table shows the weekly sales (in thousands of euros) and the test results for a random sample of eight sales representatives.

Sales representative	A	B	C	D	E	F	G	H
Weekly sales	10	12	28	24	18	16	15	12
Test results	55	60	85	75	80	85	65	60

a) Calculate the linear weekly function parameters of a week depending on the prediction test result.

b) Interpret the estimated regression function slope.

c) What is the estimation for weekly sales if a person would score 90 points on the test?

EXERCISE 4.

a) Calculate the linear weekly function parameters of a week depending on the prediction test result.

n = 8

Sales representative	Weekly sales (Yi)	Test results (Xi)	Xi^2	Xi*Yi
A	10	55	3025	550
B	12	60	3600	720
C	28	85	7225	2380
D	24	75	5625	1800
E	18	80	6400	1440
F	16	85	7225	1360
G	15	65	4225	975
H	12	60	3600	720
∑	**135**	**565**	**40925**	**9945**

$$8\,\widehat{\beta_0} + 565\widehat{\beta_1} = 135 \;\; / \times(-70{,}625)$$

$$565\widehat{\beta_0} + 40.925\,\widehat{\beta_1} = 9.945$$

$$-565\,\widehat{\beta_0} - 39.903{,}125\,\widehat{\beta_1} = -9.534{,}375$$

$$565\,\widehat{\beta_0} + 40.925\,\widehat{\beta_1} = 9.945$$

$$1.021{,}875\,\widehat{\beta_1} = 410{,}625 \;\; / : 1.021{,}875$$

$$\widehat{\beta_1} = 0{,}4018$$

$$8\,\widehat{\beta_0} + 565 \times 0{,}4018 = 135$$

$$\widehat{\beta_0} = -11{,}5046$$

EXERCISE 4.

a) Calculate the linear weekly function parameters of a week depending on the prediction test result.

n = 8

Sales representative	Weekly sales (Yi)	Test results (Xi)	Xi^2	Xi*Yi
A	10	55	3025	550
B	12	60	3600	720
C	28	85	7225	2380
D	24	75	5625	1800
E	18	80	6400	1440
F	16	85	7225	1360
G	15	65	4225	975
H	12	60	3600	720
∑	**135**	**565**	**40925**	**9945**

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \begin{bmatrix}8 & 565\\ 565 & 40.925\end{bmatrix}^{-1} \times \begin{bmatrix}135\\ 9.945\end{bmatrix}$$

$$\begin{bmatrix}8 & 565\\ 565 & 40.925\end{bmatrix}^{-1} = \frac{1}{|detA|} \times A^* = \frac{1}{|8 \times 40.925 - 565 \times 565|} \times \begin{bmatrix}40.925 & -565\\ -565 & 8\end{bmatrix}$$

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \frac{1}{|8.175|} \times \begin{bmatrix}40.925 & -565\\ -565 & 8\end{bmatrix} \times \begin{bmatrix}135\\ 9.945\end{bmatrix}$$

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \frac{1}{|8.175|} \times \begin{bmatrix}40.925 \times 135 - 565 \times 9.945\\ -565 \times 135 + 8 \times 9.945\end{bmatrix}$$

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \frac{1}{|8.175|} \times \begin{bmatrix}-94.050\\ 3.285\end{bmatrix} \qquad \begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \begin{bmatrix}-11.5046\\ 0,4018\end{bmatrix}$$

EXERCISE 4.

b) Interpret the estimated regression function slope.

$$\widehat{Y}_i = -11{,}5046 + 0{,}4018X_i$$

Each score of 1 on the test achieved by a sales representative should yield, on average, $ 401.8 in weekly sales revenue [$ 0.4018 * $ 1,000].

c) What is the estimation for weekly sales if a person would score 90 points on the test?

$$X_i = 90\ bodova$$

$$\widehat{Y}_i = -11{,}5046 + 0{,}4018X_i \quad \rightarrow \quad \widehat{Y}_i = -11{,}5046 + 0{,}4018 \times 90 = 24{,}6574$$

The person who scored 90 on the test is expected to have an average weekly sales revenue of $ 24,657.40 [$ 24.6574 * $ 1,000].

EXERCISE 5

EXERCISE 5.

- Which of these models are correctly labelled?

$Y_i = \alpha_0 + \alpha_1 X_i + u_i$	CORRECT
$Y_i = \hat{\alpha}_0 + \hat{\alpha}_1 X_i + e_i$	CORRECT
$Y_i = \hat{\alpha}_0 + \hat{\alpha}_1 X_i + u_i$	FALSE
$\hat{Y}_i = \alpha_0 + \alpha_1 X_i$	FALSE
$\hat{Y}_i = \alpha_0 + \alpha_1 X_i + e_i$	FALSE
$\hat{Y}_i = \hat{\alpha}_0 + \hat{\alpha}_1 X_i + e_i$	FALSE

Thank you!

Questions?

Linear regression models; Statistical significance

Exercises 7.

International Business / 2020

ILO2	test the statistical significance of the regression parameters and the applied regression model and to economically interpret the results obtained with the help of available program tools

Linear regression models

- Multiple regression models
- Statistical significance of regression models: Analysis of variance, Coefficient of determination, Standard error of regression, F-test
- Statistical significance of regression variables: t-test, p-test
- Hypothesis testing

Statistical significance

- Statistical significance of regression variables :
  1. T-test (critical values table)
  2. P-test

- Statistical significance of regression models :
  1. F-test (critical values table)

Statistical significance

- Coefficient of determination :
  A. Normal: R^2
  → explains how many events / variations of the dependent variable are covered by the model (in %)
  → growing with each added variable

  B. Adjusted: $\overline{R^2}$
  → explains how many events / variations of the dependent variable are covered by the model (in %)
  →growing only if a significant variable is added to the model
  →falling if a insignificant variable is added to the model
  → **more accurate** than the normal R^2

EXERCISE 1

EXERCISE 1.

- On the sample of 10 primary school students, the relationship between body weight and the monthly pocket money was examined.

Pupil number	Monthly pocket money (kn)	Body weight
1	300	64
2	500	60
3	120	54
4	1000	48
5	400	80
6	300	50
7	200	52
8	350	46
9	600	42
10	150	52

EXERCISE 1.

a) Determine the degree of correlation between these two variables and explain its meaning.

b) Test the significance of the independent variable pocket money in regards with significance level 5 %.

c) Test the significance of the regression model in regards with significance level 5 %.

d) Goodness to fit: solve the ANOVA table. Determine the coefficients of determination. Compute the standard error of the regression. Interpret the results.

EXERCISE 1.

SUMMARY OUTPUT

s(ß) = Coef(ß) / t(ß)

	Coefficients	Standard Error	t Stat	P-value
Intercept	57,76578106	6,748028741		2,67383E-05
Monthly pocket money (kn)		0,014555204	-0,519798137	0,617276862

Coef(ß) = s(ß)*t(ß) t(ß) = Coef(ß) / s(ß)

ANOVA

	df	SS	MS	F	Significance F
Regression	k	ESS	ESS/k	0,270190103	0,617276862
Residual	n-k-1	RSS 1038,5251	RSS/n-k-1	$\frac{ESS/k}{RSS/n-k-1}$	
Total	n-1	TSS	TSS/n-1		

Regression Statistics	
Multiple R	0,180749445
R Square	
Adjusted R Square	
Standard Error	
Observations	

R2 = $\frac{ESS}{TSS}$

R2 adj = 1 - $\frac{ESS/k}{RSS/n-k-1}$

s = √RSS/n-k-1

n

=T.INV.2T()

=F.INV.RT()

EXERCISE 1.

SUMMARY OUTPUT

s(ß) = Coef(ß) / t(ß)

	Coefficients	*Standard Error*	*t Stat*	*P-value*
Intercept	57,76578106	6,748028741	8,560393453	2,67383E-05
Monthly pocket money (kn)	-0,007565768	0,014555204	-0,519798137	0,617276862

Coef(ß) = s(ß)*t(ß)

t(ß) = Coef(ß) / s(ß)

ANOVA

		df		*SS*		*MS*	*F*	*Significance F*
Regression	k	1	ESS	35,07490045	ESS/k	35,07490045	0,270190103	0,617276862
Residual	n-k-1	8	RSS	1038,5251	RSS/n-k-1	129,8156374	$\frac{ESS/k}{RSS/n-k-1}$	
Total	n-1	9	TSS	1073,6	TSS/n-1			

Regression Statistics	
Multiple R	0,180749445
R Square	0,032670362
Adjusted R Square	-0,088245843
Standard Error	11,39366655
Observations	10

$R2 = \frac{ESS}{TSS}$

$R2\ adj = 1 - \frac{ESS/k}{RSS/n-k-1}$

$s = \sqrt{RSS/n-k-1}$

n

=T.INV.2T()

=F.INV.RT()

EXERCISE 1.

- Determine the degree of correlation between these two variables and explain its meaning.

$$\hat{Y}_i = 57{,}7658 - 0{,}0076X_i$$

- INTERPRETATION:

If the monthly pocket money increases by 1 kn, in average the body weight of a child would decrease by 0,0076 kg.

EXERCISE 1.

- Test the significance of the independent variable pocket money in regards with significance level 5 %.

$$\alpha = 0{,}05$$ Significance level

$$n = 10$$ Number of observations (10 children)

$$k = 1$$ Number of independent variables (monthly pocket money)

$$df = n - k - 1 = 8$$ Degrees of freedom

$$t_c = 2{,}306$$ → From the table!

EXERCISE 1.

T-TEST:

1. Hypothesis: $H_0: \ \beta_1 = 0$

   $H_A: \ \beta_1 \neq 0$

2. Testing:

$$t_{\widehat{\beta_1}} = \frac{\widehat{\beta_1}}{s_{\beta_1}} = \frac{-0{,}007565768}{0{,}014555204} = -0{,}5198$$

$p: \quad 0{,}6173 > 0{,}05$

$|t_{\beta_1}| > t_c$

$|-0{,}5198| < 2{,}306$

→ we conclude H_0!

Interpretation:

With significance level od 5% we can choose hypothesis Ho and conclude that the monthly pocket money does not significantly affect a childs body weight.

Testing of independent variables:

T- test

- $|t_\beta| < t_c$ → WE CHOOSE H_0! → the variable is statistically insignificant (not important) for the model
- $|t_\beta| \geq t_c$ → WE CHOOSE H_A! → the variable is statistically significant (important) for the model

P- test

- $p > \alpha$ → WE CHOOSE H_0! → the variable is statistically insignificant (not important) for the model
- $p < \alpha$ → WE CHOOSE H_A! → the variable is statistically significant (important) for the model

EXERCISE 1.

- Test the significance of the regression model in regards with significance level 5 %.

$\alpha = 0{,}05$ Significance level

$n = 10$ Number of observations (10 children)

$k = 1$ Number of independent variables (monthly pocket money)

$df = n - k - 1 = 8$ Degrees of freedom

$F_c = 5{,}318$ → From the table!

EXERCISE 1.

F-TEST:

1. Hypothesis: $H_0: \quad \beta_1 = \beta_2 = 0$

   $H_A: \quad H_o \; in \; not \; correct$

2. Testing: $F = \frac{ESS/k}{RSS/(n-k-1)} = \frac{35{,}07490045/1}{129{,}8156374} = 0{,}2702$

   $F < F_c$

   $0{,}2702 < 5{,}318$

   → we choose H_0!

Interpretation:

With significance level od 5% we can choose hypothesis Ho and conclude that the model is not statistically important.

EXERCISE 1.

- Goodness to fit: solve the ANOVA table. Determine the coefficients of determination. Compute the standard error of the regression. Interpret the results.

- $R^2 = \frac{ESS}{TSS} = 0{,}0327$ → the closet to 1 the better

0,8 and more = good for „t" ; 0,6 and more = good for „i"

- 3,27 % of the variance of the dependent variable is explained with this model.

EXERCISE 1.

- Goodness to fit: solve the ANOVA table. Determine the coefficients of determination. Compute the standard error of the regression. Interpret the results.

- $\overline{R^2} = 1 - \frac{RSS/n-k-1}{TSS/n-1} = -0{,}0882$ → the closet to 1 the better

0,8 and more = good for „t” ; 0,6 and more = good for „i”

EXERCISE 1.

- Goodness to fit: solve the ANOVA table. Determine the coefficients of determination. Compute the standard error of the regression. Interpret the results.

$$s = \sqrt{\frac{RSS}{n-k-1}} = 11{,}3937$$

- The standard error of the model is 11,39 kg.

EXERCISE 1.

- Goodness to fit: solve the ANOVA table. Determine the coefficients of determination. Compute the standard error of the regression. Interpret the results.

- The model is not a good fit to reality of the model. The coefficients of determination are both low, and the standart error is high.

EXERCISE 2

EXERCISE 2.

- The rankings on the classification exam and the average grade (GPA) during the studies for 20 students of psychology are set (excel file “Exercises 7”).

a) Determine the equation of the linear regression model that shows the dependence of the classification obtained and explain the meaning of the obtained parameters.

b) Test the significance of the independent variable GPA in regards with significance level 1 %.

c) Test the significance of the regression model in regards with significance level 1 %.

d) Goodness to fit: solve the ANOVA table. Determine the coefficients of determination. Compute the standard error of the regression. Interpret the results.

EXERCISE 2.

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0,31249051
R Square	0,097650319
Adjusted R Square	0,047519781
Standard Error	10,15657985
Observations	20

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	1	200,9399442	200,9399442	1,947920836	0,179790849
Residual	18	1856,810056	103,1561142		
Total	19	2057,75			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	36,88381089	12,13253121	3,040075501	0,007043235	11,39430867	62,37331311
Prosjek	-4,616655812	3,307819827	-1,395679346	0,179790849	-11,56612739	2,332815768

EXERCISE 2.

- Determine the equation of the linear regression model that shows the dependence of the classification obtained and explain the meaning of the obtained parameters.

$$\hat{Y}_i = 36{,}8838 - 4{,}6167X_i$$

- Interpretation:

If the achieved average score increases by 1, a drop in the ranking list of 4.6167 positions is expected. The economic criterion is not met.

EXERCISE 2.

- Test the significance of the independent variable GPA in regards with significance level 1 %.

$\alpha = 0{,}01$

$n = 20$

$k = 1$

$df = n - k - 1 = 18$

$t_c = 2{,}878$

T-TEST:

1. Hypothesis: $H_0: \quad \beta_1 = 0$

   $H_A: \quad \beta_1 \neq 0$

2. Testing:

$t_{\widehat{\beta_1}} = -1{,}3957$

$p: \quad 0{,}1798 > 0{,}05$

$|t_{\beta_1}| < t_c$

$|-1{,}3957| < 2{,}878$

$\rightarrow H_0!$

Interpretation:

With a significance level of 1%, we accept the Ho hypothesis and conclude that the achieved GPA does not statistically significantly affect the student's rank.

EXERCISE 2.

- Test the significance of the regression model in regards with significance level 1 %.

$\alpha = 0{,}01$

$n = 20$

$k = 1$

$df = n - k - 1 = 18$

$F_c = 8{,}285$

F-TEST:

1. Hypothesis: $H_0\colon\ \beta_1 = 0$

   $H_A\colon\ H_o\ is\ false$

2. Testing: $F = 1{,}9479$

   $F < F_c$

   $1{,}9479 < 8{,}285$

   → H_0!

Interpretation:

With a significance level of 1%, we accept the Ho hypothesis and conclude that the model is not statistically significant.

EXERCISE 2.

- Goodness to fit: solve the ANOVA table. Determine the coefficients of determination. Compute the standard error of the regression. Interpret the results.

$R^2 = 0{,}0977$ 9.77% of the variance of the dependent variable was explained by the model.

$\overline{R^2} = 0{,}0475$

$s = 10{,}1566$

The standard error of the model is 10.16 positions on the ranking scale.

The model fit to reality is poor. The coefficients of determination are low and the standard error of the model is high. The rank of the students was obviously much more influenced by some other variable that was not included in the model (eg the result achieved at the entrance exam).

EXERCISE 3

EXERCISE 3.

- The data on investment in marketing and the annual profit realized in 15 tourist agencies are given. (excel file “Exercises 7”).

a) Determine the equation of the linear regression model that shows the dependence of the marketing investment profit and explain the meaning of the obtained parameters.

b) Test the significance of the independent variable pocket money in regards with significance level 5 %.

c) Test the significance of the regression model in regards with significance level 1 %.

d) Goodness to fit: solve the ANOVA table. Determine the coefficients of determination. Compute the standard error of the regression. Interpret the results.

EXERCISE 3.

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0,984591461
R Square	0,969420344
Adjusted R Square	0,967068063
Standard Error	22,15320003
Observations	15

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	1	202253,3978	202253,3978	412,1192382	3,14609E-11
Residual	13	6379,935532	490,7642717		
Total	14	208633,3333			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	16,29939719	11,19519937	1,455927371	0,169135698	-7,886360629	40,485155
Ulaganje u marketing (tisuće €)	2,839640545	0,139878808	20,30072014	3,14609E-11	2,537450751	3,141830338

EXERCISE 3.

- Determine the equation of the linear regression model that shows the dependence of the marketing investment profit and explain the meaning of the obtained parameters.

$$\hat{Y}_i = 16{,}2994 + 2{,}8396X_i$$

- <u>Interpretation:</u>

If the amount of investment in marketing increases by € 1,000, the average annual profit is expected to increase by € 2,839.60.

EXERCISE 3.

- Test the significance of the independent variable pocket money in regards with significance level 5 %.

$\alpha = 0{,}05$

$n = 15$

$k = 1$

$df = n - k - 1 = 13$

$t_c = 2{,}160$

T-TEST:

1. Hypothesis: $H_0: \quad \beta_1 = 0$

   $H_A: \quad \beta_1 \neq 0$

2. Testing:

$$t_{\widehat{\beta_1}} = 20{,}3007$$

$$p: \quad 0{,}0000 < 0{,}05$$

$$|t_{\beta_1}| > t_c$$

$$|20{,}3007| > 2{,}160$$

$$\rightarrow H_A!$$

Interpretation:

With a significance level of 5%, we accept hypothesis Ha and conclude that investments in marketing have a statistically significant effect on the company's annual profit.

EXERCISE 3.

- Test the significance of the regression model in regards with significance level 1 %.

$\alpha = 0{,}01$

$n = 15$

$k = 1$

$df = n - k - 1 = 13$

$F_c = 9{,}074$

F-TEST:

1. Hypothesis: H_0: $\beta_1 = 0$

   H_A: $H_o\ is\ false$

2. Testing: $F = 412{,}1192$

   $F > F_c$

   $412{,}1192 > 9{,}074$

   → H_A!

Interpretation:

With a significance level of 1%, we accept hypothesis Ha and conclude that the model is statistically significant.

EXERCISE 3.

- Goodness to fit: solve the ANOVA table. Determine the coefficients of determination. Compute the standard error of the regression. Interpret the results.

$R^2 = 0{,}9694$ 96.94% of the variance of the dependent variable was explained by the model.

$\overline{R^2} = 0{,}96707$

$s = 22{,}1532$

The standard error of the model is € 22.15.

The model fit to reality is high. The coefficients of determination are high and the standard error of the model is low.

EXERCISE 4

EXERCISE 4.

- Data on annual profit (Y), investment in marketing (X1), investment in employee education (X2) and cost per product unit (X3) are set in 30 similar-profile companies (excel file “Exercises 7”).

a) Determine the equation of the multiple linear regression model.

b) Explain the meaning of the parameters.

c) Assess the statistical significance of the model.

d) Estimate the statistical significance of the regression coefficients (regression variables).

e) Based on the regression model obtained, evaluate the annual profit of the factory, which would invest 1.5 million kn both into marketing as well as in education of employees, and the costs per unit of product decreased to 40 kn.

EXERCISE 4.

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0,899950961
R Square	0,809911732
Adjusted R Square	0,78797847
Standard Error	8,372220377
Observations	30

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	3	7764,920742	2588,306914	36,92618741	1,60649E-09
Residual	26	1822,445925	70,09407403		
Total	29	9587,366667			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	73,80371603	24,2971764	3,037542915	0,005371709	23,86015466	123,7472774
X1	0,038160753	0,016047809	2,37794159	0,025046809	0,005174009	0,071147497
X2	0,013431784	0,014124941	0,950926728	0,350398289	-0,015602448	0,042466016
X3	-0,599829653	0,27976297	-2,144063792	0,041549866	-1,174890673	-0,024768633

EXERCISE 4.

- Determine the equation of the multiple linear regression model. Explain the meaning of the parameters.

$$\hat{Y}_i = 73{,}8037 + 0{,}0382X_{1i} + 0{,}1343X_{2i} - 0{,}5998X_{3i}$$

- Interpretation:

*If the amount of investment in marketing increases by HRK 1,000, the average annual profit is expected to increase by HRK 38,160.75, cet. par. (0,038160753*1.000.000 kn)*

*If the amount of investment in employee education increases by HRK 1,000, the average annual profit is expected to increase by HRK 13,431.78, cet. par. (0,013431784*1.000.000 kn)*

*If the amount of unit production costs increases by HRK 1, the average annual profit is expected to decrease by HRK 599,829.65, cet. par. (0,599829653*1.000.000 kn)*

EXERCISE 4.

- Assess the statistical significance of the model.

$\alpha = 0{,}05$

$n = 30$

$k = 3$

$df = n - k - 1 = 26$

$F_c = 2{,}975$

F-TEST:

1. Hypothesis: $H_0\text{:}\ \ \beta_1 = \beta_2 = \beta_3 = 0$

   $H_A\text{:}\ \ H_o\ is\ false$

2. Testing: $F = 36{,}9262$

   $F > F_c$

   $36{,}9262 > 2{,}975$

   → H_A!

Interpretation:

With a significance level of 1%, we accept hypothesis Ha and conclude that the model is statistically significant.

$\alpha = 0{,}05$

$n = 30$

$k = 3$

$df = n - k - 1 = 26$

$t_c = 2{,}056$

EXERCISE 4.

- Estimate the statistical significance of the regression coefficients (regression variables).

T-TEST:

1. Hypothesis:

$H_0: \ \beta_1 = 0$ $\quad H_A: \ \beta_1 \neq 0$

$H_0: \ \beta_2 = 0$ $\quad H_A: \ \beta_2 \neq 0$

$H_0: \ \beta_3 = 0$ $\quad H_A: \ \beta_3 \neq 0$

2. Testing:

$t_{\widehat{\beta_1}} = 2{,}3779$

$|t_{\beta_1}| > t_c$

$|2{,}3779| > 2{,}056$

$\rightarrow H_A!$

Interpretation:

With a significance level of 5%, we accept hypothesis Ha and conclude that investments in marketing have a statistically significant effect on the company's annual profit.

$t_{\widehat{\beta_2}} = 0{,}9509$

$|t_{\beta_2}| < t_c$

$|0{,}9509| < 2{,}056$

$\rightarrow H_0!$

Interpretation:

With a significance level of 5%, we accept the Ho hypothesis and conclude that investments in employee education do not statistically significantly affect the company's annual profit.

$t_{\widehat{\beta_3}} = -2{,}1441$

$|t_{\beta_3}| > t_c$

$|-2{,}1441| > 2{,}056$

$\rightarrow H_A!$

Interpretation:

With a significance level of 5%, we accept Hypothesis Ha and conclude that unit production costs statistically significantly affect the company's annual profit.

EXERCISE 4.

- Based on the regression model obtained, evaluate the annual profit of the factory, which would invest 1.5 million kn both into marketing as well as in education of employees, and the costs per unit of product decreased to 40 kn.

$$\hat{Y}_i = 73{,}8037 + 0{,}0382X_{1i} + 0{,}1343X_{2i} - 0{,}5998X_{3i}$$

$$\hat{Y}_i = 73{,}8037 + 0{,}0382 \times .1500 + 0{,}1343 \times 1.500 - 0{,}5998 \times 40 = 127{,}19933496$$

$$X_1 = \frac{1.500.000\ kn}{1.000\ kn} = 1.500$$

$$X_2 = \frac{1.500.000\ kn}{1.000\ kn} = 1.500$$

$$X_3 = 40\ kn$$

$$\hat{Y}_i = 127{,}19933496 * 1.000.000\ kn = \mathbf{127.199.334{,}96\ kn}$$

How to?

- In detail guide how to do multiple regression in excel: https://www.youtube.com/watch?v=e7PtveMRMbs

Thank you!

Questions?

Non-linear regression models; Dummy variables

Exercises 8.

International Business / 2020

ILO2	test the statistical significance of the regression parameters and the applied regression model and to economically interpret the results obtained with the help of available program tools

Non-linear regression models

- For example:
  - we will not spend the same amounts of money at certain levels of revenue on certain things
  - each next cake does not carry the same wish as the previous one

  → varialbes in logarithms (ln)

  → change in %

Non-linear regression models

Interpretation:

1. Linear model: $Y = \beta_0 + \beta_1 X$

→ if X increases for 1 unit (from X), Y changes for β_1 units (from Y)

2. Exponential model: $\ln Y = \beta_0 + \beta_1 \ln X$

→ if X increases by 1%, Y increases by β_1 %

3. Semi-log model:

a) $\ln Y = \beta_0 + \beta_1 X$

→ if X increases by 1, Y changes for β_1 x 100%

4. Reciprocal model
5. Polynomial model

b) $Y = \beta_0 + \beta_1 \ln X$

Dummy variables

- Binary, qualitative or yes / no variables

→ are used to mark the existence or absence of a certain qualitative phenomenon

→ **1** = the event happened

→ **0** = the event didn't happen

EXERCISE 1

EXERCISE 1.

- Estimate the impact on sales of retail market trade for the following: population in the trade area, number of competing shops in the area, location (city centre / periphery).

- The data:
- Y_i – annual sales revenue of the i^{th} shop (in houndred thousands \$)
- X_{1i} – population within 1 km from the i^{th} shop (in thousands)
- X_{2i} – number of competing shops within 1 km from the i^{th} shop
- X_{3i} – location – city centre of periphery (Remark: The first 6 stores in the sample were in the centre of the city, the remaining on the periphery)

EXERCISE 1.

- i = 1, ..., 9.

i	Y	X_1	X_2	X_3
1	10	15	23	1
2	15	32	30	1
3	20	48	36	1
4	12	18	13	1
5	25	35	9	1
6	23	40	16	1
7	17	30	5	0
8	16	26	12	0
9	20	40	15	0

EXERCISE 1.

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0,966533348
R Square	0,934186714
Adjusted R Square	0,894698742
Standard Error	1,598899794
Observations	9

ANOVA

	df	SS	MS	F	Significance F
Regression	3	181,4398195	60,47993982	23,65750046	0,002209768
Residual	5	12,78240276	2,556480551		
Total	8	194,2222222			

	Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%
Intercept	6,024448593	1,92764264	3,125293282	0,026095722	1,069285437	10,97961175
X1	0,472692776	0,057138135	8,272807276	0,000421061	0,325814525	0,619571028
X2	-0,326620385	0,07083939	-4,610717054	0,005783831	-0,508718835	-0,144521935
X3	3,577975893	1,361797382	2,627392254	0,046679751	0,077364278	7,078587508

EXERCISE 1.

a) Type of statistical data?

Cross-section data.

b) How to introduce variable data for X_3?

Dummy variables: 1 = city centre

0 = periphery

c) Construct a multiple linear regression model based on data for all the specified variables? Evaluate the results.

$$\widehat{Y}_i = 6{,}0244 + 0{,}4727X_{1i} - 0{,}3266X_{2i} + 3{,}5780X_{3i}$$

EXERCISE 1.

d) What are the meanings of the parameters value with a particular explanatory variable (are the predictions in line with expectations)?

If the population within 1 km range from the shop would increase by 1.000 people, in average, the annual revenue of the shop would increase by 47.269,28$, cet. par.

(0,472692776 x 100.000 $)

If the number of competing shops within 1 km radius from the shop would increase by 1, in average, the annual revenue of the shop would decrease by 32.662,20$, cet. par.

(-0,326620385 x 100.000 $)

A shop which is located at the city centre has in average a 357.797,59$ higher annual revenue than a shop located on the periphery, cet. par.

(3,577975893 x 100.000 $)

EXERCISE 1.

e) Evaluate with the model how much annual revenue would an average trade be located outside the centre of the city in the area of population of only 8.000 inhabitants in 1 km circuit and if there is only one competing market in that area?

$$\hat{Y}_i = 6{,}0244 + 0{,}4727X_{1i} - 0{,}3266X_{2i} + 3{,}5780X_{3i}$$

$X_{1i} = 8$ (because it already is in thousands)
$X_{2i} = 1$ (one competing shop)
$X_{3i} = 0$ (on the periphery)

$$\hat{Y}_i = 6{,}0244 + 0{,}4727 \times 8 - 0{,}3266 \times 1 + 3{,}5780 \times 0$$
$$\hat{Y}_i = 9{,}4794$$

The annual revenue of this particular shop is 947.940 $. (9,4794 x 100.000$)

EXERCISE 2

EXERCISE 2.

- The following regression model was evaluated on a sample of 125 employees using the least squares method (standard parameter errors are in brackets):

- $$\hat{HPL}_i = 5.00 + 0.5GOD_i - 0.01GOD^2{}_i - 3.00FEM_i - 2.5PART_i$$

  (2.00) (0.10) (0.002) (2.00) (3.25)

  TSS=1000

  RSS=200

- The variables are:

  HPL – paid hours of work (u $)

  GOD – years of employee experience

  GOD2 – squared years of employee experience

  FEM – gender: FEM = 1 if the employee is female

  FEM = 0 if the employee is not female

  PART – employment: PART = 1 if the employee works part-time

  PART = 0 if the employee works full-time

EXERCISE 2.

a) Interpret the estimated parameters with dummy variables.

If the employee is female, in average, their paid working hour will be 3$ less then a male employee, cet. par.

If the employee works part-time, in average, their pair hour of work will be 2,5$ less then the working hour of a full-time working employee, cet. par.

b) Determine the level of work experience on which the paid work hour is highest.

We use partial derivations, so that all of tha variables which do not include the variable GOD are threated as constants. Hence, all of the non-GOD variables will be 0.

$$\frac{dHPL_i}{dGOD_i} = 0 \qquad \frac{d(5{,}00+0{,}5GOD-0{,}01GOD^2-3{,}00FEM-2{,}5PART)_i}{dGOD_i} = 0$$

$$\left(0{,}5GOD_i - 0{,}01GOD_i^2\right)' = 0 \qquad 0{,}5GOD_i - 2\times 0{,}01GOD_i$$

$$0{,}5 - 0{,}02GOD_i = 0$$

EXERCISE 2.

c) Test the hypothesis that the price of a working hour does not depend on gender (with 95% confidence).

$$\alpha = 0{,}05$$
$$n = 125$$
$$k = 4$$
$$df = n - k - 1 = 120$$
$$t_c = 1{,}980$$

With 95% confidence level we choose hypothesis H0 and can conclude that the variable FEM, or gender of the employee, does not significantly affect their paid working hour.

T-TEST:

1. HYPOTHESIS: $H_0\colon\ \beta_3 = 0$

   $H_A\colon\ \beta_3 \neq 0$

2. TESTING:

$$t_{\widehat{\beta_3}} = \frac{\widehat{\beta_3}}{s_{\beta_3}} = \frac{-3{,}00}{2{,}00} = -1{,}5$$

$$|t_{\beta_1}| < t_c$$

$|-1{,}5| < 1{,}980$ → choosing H_0!

EXERCISE 2.

d) Calculate the determination coefficient (ordinary and adjusted). What can be deduced based on their values?

$$R^2 = \frac{ESS}{TSS} = \frac{800}{1.000} = 0,8$$

→ 80 % of the variance of the dependent variable is explained by the model!

$$\overline{R^2} = 1 - \frac{RSS \div (n-k-1)}{TSS \div (n-1)} = 1 - \frac{200 \div 120}{1.000 \div 124} = 0,7933$$

EXERCISE 2.

e) Test the statistical significance of the entire regression model (with significance 5%).

$$\alpha = 0{,}05$$
$$n = 125$$
$$k = 4$$
$$df = n - k - 1 = 120$$
$$F_c = 2{,}45$$

With 5% significance level we choose hypothesis HA and can conclude that the model for paid working hours of employees is statistically important.

F-TEST:

1. HYPOTHESIS: $H_0{:}\ \widehat{\beta_1} = \widehat{\beta_2} = \widehat{\beta_3} = \widehat{\beta_4} = 0$

   $H_A{:}\ H_0\ is\ not\ correct$

2. TESTING: $F = \frac{ESS/k}{RSS/(n-k-1)} = \frac{800/4}{200/120} = 120$

   $F > F_c$

   $120 > 2{,}45$

   → declining H_0, choosing H_A!

EXERCISE 2.

f) If the variable GOD^2 was excluded from the model, would the model's conformability be increased or decreased?

$$t_{\widehat{\beta_2}} = \frac{\widehat{\beta_2}}{s_{\widehat{\beta_2}}} = \frac{-0{,}01}{0{,}002} = -5$$

Example:

$\alpha = 0{,}05$	$\alpha = 0{,}01$
$tc = 1{,}980$	$tc = 2{,}617$
$\lvert -5 \rvert > 1{,}980$	$\lvert -5 \rvert > 2{,}617$
→ $\boldsymbol{H_A}$	→ $\boldsymbol{H_A}$ → important variable

→ The model's conformability would decrease, because we would exclude a statistically important variable from the model.

EXERCISE 2.

g) If the variable PART would be excluded from the model, would the modality adaptation of the observations be increased or decreased?

$$t_{\widehat{\beta_4}} = \frac{\widehat{\beta_4}}{s_{\widehat{\beta_4}}} = \frac{-2{,}5}{3{,}25} = -0{,}7692$$

Example:

	$\alpha = 0{,}05$	$\alpha = 0{,}01$	
	$tc = 1{,}980$	$tc = 2{,}617$	
	$\lvert -0{,}7692 \rvert < 1{,}980$	$\lvert -0{,}7692 \rvert < 2{,}617$	
	$\rightarrow \boldsymbol{H_0}$	$\rightarrow \boldsymbol{H_0}$	→ not important variable

→ The model's conformability would increase, because we would exclude a statistically non-important variable from the model.

EXERCISE 3

EXERCISE 3.

Y = house prices in thousands of € (for m2)

X = income in thousands of €

- Interpret the following regression models:

a) $Y = 10 + 0,3x$ $R2 = 0,50$

b) $\ln Y = 9,0+0,5\ln x$ $R2 = 0,60$

c) $\ln Y = 8,5+0,04x$ $R2 = 0,40$

d) $Y = 11 + 20\ln x$ $R2 = 0,70$

*see: Slide no. 3

EXERCISE 3.

Y = house prices in thousands of € (for m2)

X = income in thousands of €

a) Y = 10 + 0,3x R^2 =0,50

If our income would increase by 1.000 €, the house prices would increase by 300 €.

(0,3*1.000 €)

EXERCISE 3.

Y = house prices in thousands of € (for m2)

X = income in thousands of €

b) $\ln Y = 9{,}0 + 0{,}5 \ln x$ $R^2 = 0{,}60$

If our income would increase by 1 %, the house prices would increase by 0,5 %.

EXERCISE 3.

Y = house prices in thousands of € (for m2)

X = income in thousands of €

c) $\ln Y = 8{,}5+0{,}04x$ $R^2 = 0{,}40$

If our income would increase by 1.000 €, the house prices would increase by 4 %.
(0,04*100%)

EXERCISE 3.

Y = house prices in thousands of € (for m2)

X = income in thousands of €

d) $Y = 11 + 20\ln x$ $R^2 = 0{,}70$

If our income would increase by 1 %, the house prices would increase by 200 €.

(20*1.000 € = 20.000 € / 100 units (€) = 200 €)

Thank you!

Questions?

Regression analysis in Excel; Preparation for the exam

Exercises 9.

International Business

ILO2	test the statistical significance of the regression parameters and the applied regression model and to economically interpret the results obtained with the help of available program tools

Continuous assessment 2

TASK NR.	DESCRIPTION	POINTS
1.	**SIMPLE LINEAR REGRESSION** a) Based on the given data estimate (calculate) the linear regression model [4 points] b) Correctly state the interpretation of the model; is the economical criteria met? [3 points] c) Make a prediction based on the estimated model [2 points]	9
2.	**MULTIPLE REGRESSION MODEL** a) Fill in the dummy variables [1 point] b) Make the regression analysis using Excel, write down the model and its intepretations [5 points] c) Make a prediction using the model and write down the interpretation [2 points] d) Test the significance of an independent variable [3 points] e) Goodness to fot: analyse the statistical significance of the model (R2, R2adj, S, V, F-test) [5 points]	16
	TOTAL:	**25**

EXERCISE 1

Exercise 1

- The table provides data on average monthly temperatures by months for 2019 and data on average gas consumption in households in cubic meters.

Month	Gas consumption (m3)	Average temperature (°C)
January	4,4	6
February	5,9	-3
March	5,7	2
April	3,5	8
May	1,3	16
June	0,9	22
July	0,6	24
August	0,6	25
September	0,8	22
October	0,8	20
November	2,4	12
December	4,9	2

- Determine the regression function of the average gas consumption as a function of the average monthly temperature.
- Write down the regression function, interpret its meaning and determine whether the economic criterion is met.
- Based on the model, estimate the average gas consumption at -10°C.

Exercise 1

$$\widehat{\beta_0} \times n + \widehat{\beta_1} \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} Y_i$$

$$\widehat{\beta_0} \sum_{i=1}^{n} X_i + \widehat{\beta_1} \sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n} X_i Y_i$$

- Determine the regression function of the average gas consumption as a function of the average monthly temperature.

n = 12

Month	Gas consumption (m3) (Yi)	Average temperature (°C) (Xi)	Xi^2	Xi*Yi
January	4,4	6	36	26,4
February	5,9	-3	9	-17,7
March	5,7	2	4	17,1
April	3,5	8	64	28
May	1,3	16	256	20,8
June	0,9	22	484	19,8
July	0,6	24	576	14,4
August	0,6	25	625	15
September	0,8	22	484	17,6
October	0,8	20	400	16
November	2,4	12	144	28,8
December	4,9	2	4	14,7
SUM:	**31,8**	**156**	**3086**	**200,9**

$$12\,\widehat{\beta_0} + 156\widehat{\beta_1} = 31{,}8 \quad / \times(-13)$$

$$156\widehat{\beta_0} + 3.086\,\widehat{\beta_1} = 200{,}9$$

$$-156\,\widehat{\beta_0} - 2.028\,\widehat{\beta_1} = -413{,}4$$

$$156\widehat{\beta_0} + 3.086\,\widehat{\beta_1} = 200{,}9$$

$$1.058\,\widehat{\beta_1} = -212{,}5 \quad / : 1.058$$

$$\widehat{\beta_1} = -0{,}2009$$

$$12\,\widehat{\beta_0} + 156 \times (-0{,}2009) = 31{,}8$$

$$\widehat{\beta_0} = -5{,}2617$$

Exercise 1

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \begin{bmatrix}\mathrm{n} & X_\mathrm{i}\\ X_\mathrm{i} & X_\mathrm{i}^2\end{bmatrix}^{-1} \times \begin{bmatrix}Y_\mathrm{i}\\ X_\mathrm{i}Y_\mathrm{i}\end{bmatrix}$$

- Determine the regression function of the average gas consumption as a function of the average monthly temperature.

n = 12

Month	Gas consumption (m3) Yt	Average temperature (°C) Xt	Xi^2	Xi*Yi
January	4,4	6	36	26,4
February	5,9	-3	9	-17,7
March	5,7	2	4	17,1
April	3,5	8	64	28
May	1,3	16	256	20,8
June	0,9	22	484	19,8
July	0,6	24	576	14,4
August	0,6	25	625	15
September	0,8	22	484	17,6
October	0,8	20	400	16
November	2,4	12	144	28,8
December	4,9	2	4	14,7
SUM:	**31,8**	**156**	**3086**	**200,9**

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \begin{bmatrix}12 & 156\\ 156 & 3.086\end{bmatrix}^{-1} \times \begin{bmatrix}31,8\\ 200,9\end{bmatrix}$$

$$\begin{bmatrix}12 & 156\\ 156 & 3.086\end{bmatrix}^{-1} = \frac{1}{|detA|} \times A^* = \frac{1}{|12 \times 3.086 - 156 \times 156|} \times \begin{bmatrix}3.086 & -156\\ -156 & 12\end{bmatrix}$$

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \frac{1}{|12.696|} \times \begin{bmatrix}3.086 & -156\\ -156 & 12\end{bmatrix} \times \begin{bmatrix}31,8\\ 200,9\end{bmatrix}$$

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \frac{1}{|12.696|} \times \begin{bmatrix}3.086 \times 31,8 - 156 \times 200,9\\ -156 \times 31,8 + 12 \times 200,9\end{bmatrix}$$

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \frac{1}{|12.696|} \times \begin{bmatrix}-66.794,4\\ -2.550\end{bmatrix} \qquad \begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \begin{bmatrix}5,2611\\ -0,2009\end{bmatrix}$$

Exercise 1

- Write down the regression function, interpret its meaning and determine whether the economic criterion is met.

$$\widehat{Yt} = 5{,}2611 - 0{,}2009\,Xt$$

With the increase of temperature for 1°C it is expected that the consumption of gas will decrease for 0,2009 m3.

- Based on the model, estimate the average gas consumption at -10°C.
- $Xt = -10$

$$\widehat{Yt} = 5{,}2611 - 0{,}2009 \times (-10)$$
$$\widehat{Yt} = 7{,}2701$$

With average temperature of -10°C we predict that the gas consumption will be 7,2701 m3.

EXERCISE 2

Exercise 2

- Data were collected for a sample of 8 employees of a company:

Employee no.	Monthly net salary (thousands of HRK)	Years of work experience
1	1,5	3
2	3,5	4
3	8	5
4	10	9
5	6	4
6	1	3
7	5,5	7
8	2,5	5

- Evaluate a simple linear regression model in which the amount of the employee's monthly salary is explained by the achieved level of employee experience.
- Economically interpret the parameter estimate with an independent variable.
- What is the expected monthly net salary of an employee with 15 years of work experience?

Exercise 2

$$\widehat{\beta_0} \times n + \widehat{\beta_1} \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} Y_i$$
$$\widehat{\beta_0} \sum_{i=1}^{n} X_i + \widehat{\beta_1} \sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n} X_i Y_i$$

- Evaluate a simple linear regression model in which the amount of the employee's monthly salary is explained by the achieved level of employee experience.

Yi Xi

n = 8

Employee no.	Monthly net salary (thousands of HRK)	Years of work experience	Xi^2	Xi*Yi
1	1,5	3	9	4,5
2	3,5	4	16	14
3	8	5	25	40
4	10	9	81	90
5	6	4	16	24
6	1	3	9	3
7	5,5	7	49	38,5
8	2,5	5	25	12,5
SUM:	38	40	230	226,5

$$8\,\widehat{\beta_0} + 40\,\widehat{\beta_1} = 38 \;\; / \times(-5)$$
$$40\widehat{\beta_0} + 230\,\widehat{\beta_1} = 226{,}5$$

$$-40\,\widehat{\beta_0} - 200\,\widehat{\beta_1} = -190$$
$$40\widehat{\beta_0} + 230\,\widehat{\beta_1} = 226{,}5$$

$$30\,\widehat{\beta_1} = 36{,}5 \;\; / : 30$$
$$\widehat{\beta_1} = 1{,}2167$$

$$8\,\widehat{\beta_0} + 40 \times 1{,}2167 = 36$$

$$\widehat{\beta_0} = -1{,}3333$$

Exercise 2

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \begin{bmatrix}\mathrm{n} & X_\mathrm{i}\\ X_\mathrm{i} & X_\mathrm{i}^2\end{bmatrix}^{-1} \times \begin{bmatrix}Y_\mathrm{i}\\ X_\mathrm{i}Y_\mathrm{i}\end{bmatrix}$$

- Evaluate a simple linear regression model in which the amount of the employee's monthly salary is explained by the achieved level of employee experience.

n = 8

Employee no.	Monthly net salary (thousands of HRK)	Years of work experience	Xi^2	Xi*Yi
1	1,5	3	9	4,5
2	3,5	4	16	14
3	8	5	25	40
4	10	9	81	90
5	6	4	16	24
6	1	3	9	3
7	5,5	7	49	38,5
8	2,5	5	25	12,5
SUM:	**38**	**40**	**230**	**226,5**

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \begin{bmatrix}8 & 40\\ 40 & 230\end{bmatrix}^{-1} \times \begin{bmatrix}38\\ 226{,}5\end{bmatrix}$$

$$\begin{bmatrix}8 & 40\\ 40 & 230\end{bmatrix}^{-1} = \frac{1}{|detA|} \times A^* = \frac{1}{|8 \times 230 - 40 \times 40|} \times \begin{bmatrix}230 & -40\\ -40 & 8\end{bmatrix}$$

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \frac{1}{|240|} \times \begin{bmatrix}230 & -40\\ -40 & 8\end{bmatrix} \times \begin{bmatrix}38\\ 226{,}5\end{bmatrix}$$

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \frac{1}{|240|} \times \begin{bmatrix}230 \times 38 - 40 \times 226{,}5\\ -40 \times 38 + 8 \times 226{,}5\end{bmatrix}$$

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \frac{1}{|240|} \times \begin{bmatrix}-320\\ 292\end{bmatrix} \qquad \begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \begin{bmatrix}-1{,}3333\\ 1{,}2167\end{bmatrix}$$

Exercise 2

- Economically interpret the parameter estimate with an independent variable.

$$\widehat{Yi} = -1{,}333 + 1{,}2167\ Xi$$

For each additional year of experience we expect that the monthly net salary will increase by 1.216,70 HRK.

The economic criterion is met.

- What is the expected monthly net salary of an employee with 15 years of work experience?
- $Xi = 15$

$$\widehat{Yi} = -1{,}333 + 1{,}2167 \times 15$$
$$\widehat{Yi} = 16{,}9172$$

The expected monthly net salary of an employee wiht 15 years of work experience is 16.917,20 HRK.

EXERCISE 3

Exercise 3

- Data were collected for a sample of 115 monthly rents and variables that affect the amount of that rent (Exercises 9.):

  Y: rent (USD / month),

  X1: number of persons in the apartment,

  X2: household income (USD / year),

  X3: number of rooms,

  X4: number of bedrooms,

  X5: number of parking spaces.

a) Determine the equation of the multiple linear regression model. Explain the meaning of the parameters.

b) Assess the statistical significance of the model.

c) Estimate the statistical significance of the regression coefficients (regression variables) with significance level 5%.

d) Test the statistical significance of the regression model with significance level of 1%.

e) Based on the regression model obtained, evaluate the monthly rent for an apartment for 4 people to live in, the annual household income is 300.000 USD, with 5 rooms of which 3 bedrooms. There are 3 parking spaces that come with the apartment.

Exercise 3

- Determine the equation of the multiple linear regression model. Explain the meaning of the parameters.

$$\widehat{Y}i = 1.105{,}5396 + 104{,}9305X1_i + 0{,}0034X2_i - 13{,}6853X3_i + 42{,}9913X4_i + 8{,}0206X5_i$$

- If the number of people in the apartment would increase by 1, cet. par., the average rent per month would increase by 104,93$.
- If the household income would increase by 1$, cet. par., the average rent per month would increase by 0,0034$.
- If the number of rooms in the apartment would increase by 1, cet. par., the average rent per month would decrease by 13,69$. The economic criterion is not met.
- If the number of bedrooms in the apartment would increase by 1, cet. par., the average rent per month would increase by 43,99$.
- If the number of parking spaces would increase by 1, cet. par., the average rent per month would increase by 8,02$.

Exercise 3

- Assess the statistical significance of the model.

$R^2 = 0{,}2610$

$\overline{R^2} = 0{,}2271$

$s = 510{,}99$

The coefficient of determination is 26,10%, meaning that only 26,10% variance of the dependent variable is described with the model.

The standard error of the model is 510,99$. Looking into the average rent of these apartments of 1.657$, the standard error is large, almost 31%.

The model is not a good fit to reality – it does not describe the dependent variable in a sufficiently good manner.

Exercise 3

- Estimate the statistical significance of the regression coefficients (regression variables) with significance level 5%.

T-TEST:

$\alpha = 0{,}05$

$n = 115$

$k = 5$

$df = n - k - 1 = 109$

$t_c = 1{,}982$

1. Hypothesis:

$H_0: \ \beta_1 = 0$

$H_A: \ \beta_1 \neq 0$

$H_0: \ \beta_2 = 0$

$H_A: \ \beta_2 \neq 0$

2. Testing:

$t_{\widehat{\beta_1}} = 2{,}0501$

$|t_{\beta_1}| > t_c$

$|2{,}0501| > 1{,}982$

$\rightarrow H_A!$

Interpretation:

With a significance level of 5%, we accept Hypothesis Ha and conclude the number of people statistically significantly does affect the value of monthly rent.

$t_{\widehat{\beta_2}} = 5{,}4190$

$|t_{\beta_2}| < t_c$

$|5{,}4190| > 1{,}982$

$\rightarrow H_A!$

Interpretation:

With a significance level of 5%, we accept Hypothesis Ha and conclude the household income statistically significantly does affect the value of monthly rent.

Exercise 3

- Estimate the statistical significance of the regression coefficients (regression variables) with significance level 5%.

T-TEST:

1. Hypothesis:

$H_0: \beta_3 = 0$, $H_A: \beta_3 \neq 0$

$H_0: \beta_4 = 0$, $H_A: \beta_4 \neq 0$

$H_0: \beta_5 = 0$, $H_A: \beta_5 \neq 0$

2. Testing:

$t_{\widehat{\beta_3}} = -0{,}2561$

$|t_{\beta_3}| < t_c$

$|-0{,}2561| < 1{,}982$

$\rightarrow H_0!$

Interpretation:

With a significance level of 5%, we accept Hypothesis Ho and conclude the number of rooms statistically significantly does not affect the value of monthly rent.

$t_{\widehat{\beta_4}} = 0{,}4983$

$|t_{\beta_4}| < t_c$

$|0{,}4983| < 1{,}982$

$\rightarrow H_0!$

Interpretation:

With a significance level of 5%, we accept Hypothesis Ho and conclude the number of bedrooms statistically significantly does not affect the value of monthly rent.

$t_{\widehat{\beta_5}} = 0{,}1568$

$|t_{\beta_5}| < t_c$

$|0{,}1568| < 1{,}982$

$\rightarrow H_0!$

Interpretation:

With a significance level of 5%, we accept Hypothesis Ho and conclude the number of parking spaces statistically significantly does not affect the value of monthly rent.

Exercise 3

- Test the statistical significance of the regression model with significance level of 1%.

$\alpha = 0{,}01$

$n = 115$

$k = 5$

$df = n - k - 1 = 109$

$F_c = 3{,}190$

F-TEST:

1. Hypothesis: H_0: $\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = 0$

   H_A: H_o *is not correct*

2. Testing: $F = 7{,}6986$

   $F > F_c$

   $7{,}6986 > 3{,}190$

   → H_A!

Interpretation:

With a significance level of 1%, we accept the Ha hypothesis and conclude that the model is statistically significant.

Exercise 3

- Based on the regression model obtained, evaluate the monthly rent for an apartment for 4 people to live in, the annual household income is 300.000 USD, with 5 rooms of which 3 bedrooms. There are 3 parking spaces that come with the apartment.
- $X1_i = 4$
- $X2_i = 300.000\$$
- $X3_i = 5$
- $X4_i = 3$
- $X5_i = 3$

$$\widehat{Yi} = 1.105{,}5396 + 104{,}9305 \times 4 + 0{,}0034 \times 300.000 - 13{,}6853 \times 5 + 42{,}9913 \times 3 + 8{,}0206 \times 3$$

$$\widehat{Yi} = 2.620{,}07\$$$

- Based on the given values, the rent should be 2.620,07$ a month.

EXERCISE 4

Exercise 4

- You have 150 empirical values of variables (Exercises 9.):

  Y: lease cost of real estate (in USD),

  X1: size (in square feet),

  X2: cost per square foot (in USD),

  X3: age (in years),

  X4: renovation (in years past from the last renovation),

  X5: location: 1 = city; 0 = suburbs.

- Enter the dummy variables data: use 1 if the real estate is located in the city, and 0 if its located in the suburbs.
- Make a regression model using the data. Write down the model and interpret the meaning of the values. Determine the economic criterion of the model.
- We have a real estate of 10.000 square feet, the cost per square foot is 15 $ and is located in the city. The age of the house is 10 years, and it was renovated 3 years ago. What is the expected value their house?
- Test the significance of the variable “Age” in the model. Use significance level 5 %.
- Goodness to fit: is the model good – determine and interpret the coefficients of determination and standard error of the model? Test the significance of the whole model with significance level 5 %.
- Remove the dummy variable “Renovation” from the model. What do you expect to happen? Interpret the changes. Test the significance of the new model, with same significance 5 %.

Exercise 4

- You have 150 empirical values of variables (Exercises 9.):

  Y: lease cost of real estate (in USD),

  X1: size (in square feet),

  X2: cost per square foot (in USD),

  X3: age (in years),

  X4: renovation (in years past from the last renovation),

  X5: location: 1 = city; 0 = suburbs.

- Enter the dummy variables data: use 1 if the real estate is located in the city, and 0 if its located in the suburbs.
  - **The formula to use is: (cell of reference is the cell with „City” or „Suburbs”)**
  - **=IF([choose cell of reference]=„City”;1;0)**

Exercise 4

- Make a regression model using the data. Write down the model and interpret the meaning of the values. Determine the economic criterion of the model.

$$\hat{Y}_i = -215.007{,}4559 + 16{,}8043X_{1i} + 12.697{,}1374X_{2i} - 660{,}1721X_{3i} + 39{,}9892X_{4i} + 32.652{,}8380X_{5i}$$

- INTERPRETATIONS:

If the area of the building increases by 1 ft2, the average rental costs are expected to increase by $ 16.80, cet. par.

If the cost per ft2 of real estate increases by $ 1, on average, the rental cost is expected to increase by $ 12,697.14, cet. par.

If the age of the building increases by 1 year, on average, the cost of rent is expected to decrease by $ 660.17, cet. par.

With each additional year after the last property renovation, an average rental cost is expected to increase by $ 39.99, cet. par.

If the property is located in the city, its rental cost is expected to be in average of $ 32,652.84 more than for a rural property, cet. par.

The economic criterion is met for all independent variables except for the "Reconstruction" variable.

Exercise 4

- We have a real estate of 10.000 square feet, the cost per square foot is 15 $ and is located in the city. The age of the house is 10 years, and it was renovated 3 years ago. What is the expected value their house?

$$\hat{Y}_i = -215.007{,}4559 + 16{,}8043X_{1i} + 12.697{,}1374X_{2i} - 660{,}1721X_{3i} + 39{,}9892X_{4i} + 32.652{,}8380X_{5i}$$

$$\hat{Y}_i = -215.007{,}4559 + 16{,}8043 \times \mathbf{10.000} + 12.697{,}1374 \times \mathbf{15} - 660{,}1721 \times \mathbf{10} + 39{,}9892 \times \mathbf{3} + 32.652{,}8380 \times \mathbf{1}$$

$$\hat{Y}_i = \mathbf{169.663,99}\ \$$$

Exercise 4

- Test the significance of the variable “Age” in the model. Use significance level 5 %.

$\alpha = 0{,}05$

$n = 150$

$k = 5$

$df = n - k - 1 = 144$

$t_c = 1{,}977$

T-TEST:

1. Hypothesis: $H_0\colon\ \beta_3 = 0$

   $H_A\colon\ \beta_3 \neq 0$

2. Testing:

$$t_{\widehat{\beta_3}} = -2{,}3005$$

$$|t_{\beta_3}| > t_c$$

$$|-2{,}3005| > 1{,}977$$

$$\rightarrow H_A!$$

Interpretation:

With a significance level of 5%, we accept Hypothesis Ha and conclude that the age of the building has a statistically significant effect on the annual cost of renting the property.

Exercise 4

- Goodness to fit: is the model good – determine and interpret the coefficients of determination and standard error of the model? Test the significance of the whole model with significance level 5 %.

$R^2 = 0{,}9948$ 99,48 % of variance of the dependent variable is explained by the model.

$\overline{R^2} = 0{,}9946$

$s = 41.192{,}33$ The standard error of the model is 41.192,33$ (8,08%).

The model is highly representative.

F-TEST:

$\alpha = 0{,}05$

$n = 150$

$k = 5$

$df = n - k - 1 = 144$

$F_c = 2{,}277$

1. Hypothesis: $H_0\!: \ \beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = 0$

   $H_A\!: \ H_o \text{ is not correct}$

2. Testing: $F = 5.514{,}01$

   $F > F_c$

   $5.514{,}01 > 2{,}277$

   → H_A!

Interpretation:

With a significance level of 5%, we accept Hypothesis Ha and conclude that the model is statistically significant.

Exercise 4

- Remove the variable “Renovation” from the model. What do you expect to happen? Interpret the changes. Test the significance of the new model, with same significance 5 %.

$\alpha = 0{,}05$

$n = 150$

$k = 5$

$df = n - k - 1 = 144$

$t_c = 1{,}977$

T-TEST:

1. Hypothesis: $H_0\!: \ \beta_4 = 0$

   $H_A\!: \ \beta_4 \neq 0$

2. Testing:

$t_{\widehat{\beta_4}} = 0{,}1359$

$|t_{\beta_4}| < t_c$

$|0{,}1539| < 1{,}977$

→ H_0! → statistically insignificant variable

As this is a statistically insignificant variable, it is expected that the statistical indicators of the model will improve, ie it will become more representative.

Exercise 4

- Remove the dummy variable “Renovation” from the model. What do you expect to happen? Interpret the changes. Test the significance of the new model, with same significance 5 %.
- Statistical indicators before excluding the variable:

$$R^2 = 0{,}994804083$$
$$\overline{R^2} = 0{,}994623669$$
$$s = 41.192{,}33361$$

- Statistical indicators after excluding the variable :

$R^2 = 0{,}994803417$ DECREASE

$\overline{R^2} = 0{,}994660063$ INCREASE

$s = 40.052{,}6778$ DECREASE

- The representativeness of the model has improved - the adjusted coefficient of determination increases and the value of the standard error of the model decreases.

Exercise 4

- Remove the dummy variable “Renovation” from the model. What do you expect to happen? Interpret the changes. Test the significance of the new model, with same significance 5 %.

F-TEST:

$\alpha = 0{,}05$

$n = 150$

$k = 4$

$df = n - k - 1 = 145$

$F_c = 2{,}434$

1. Hypothesis: $H_0{:}\quad \beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$

   $H_A{:}\quad H_o\ is\ not\ correct$

2. Testing: $F = 6.939{,}49$

   $F > F_c$

   $6.939{,}49 > 2{,}434$

   → H_A!

Interpretation:

With a significance level of 5%, we accept Hypothesis Ha and conclude that the model is statistically significant.

Thank you!

Questions?

Preparation for the integrated exams

Exercises 10.

International Business / 2020

ILO1	convert the practical problem of optimization into a mathematical statement as well as interpret results of applied quantitative models in business decision making
ILO2	test the statistical significance of the regression parameters and the applied regression model and to economically interpret the results obtained with the help of available program tools

Exam dates:

INTEGRATED EXAM DATES		
1st EXAM DATE	18/06/2020	11:30
2nd EXAM DATE	02/07/2020	11:30
3rd EXAM DATE	16/07/2020	11:30
4th EXAM DATE	30/07/2020	11:30
5th EXAM DATE	01/09/2020	11:30
6th EXAM DATE	15/09/2020	11:30

IMPORTANT: In order to successfully pass the course, you need to achieve at least 50 % of the possible grade points (50). Also, each student must earn at least 50 % of the possible grade points from each of the (2) intended course learning outcomes. For more details, pleas see the new presentation about the integrated exams.

All exams will be online!

University of Rijeka Faculty of Economics	**International business**
Course: **Quantitative methods for business decisions**	
Name and Surname:	
JMBAG:	**Date:**

INTEGRATED EXAM 1

Dear students,

In front of you is the integrated exam of the course Quantitative methods for business decisions. Estimated time to write the exam is 2 full hours. The exam consists of two parts: theory and exercise tasks. It is possible to achieve a total of 80 grade points. The exam assesses both learning outcomes of the course, and the distribution of points according to the learning outcomes is even.

The first part of the exam consists of 15 theoretical questions and 4 offered answers for each question. Only 1 answer is correct, and each correct answer brings 2 grade points. Additionally, there is 1 bonus question whose correct answer brings 1 grade point that can be used if the student is missing 1 point for passing the course or 1 point for a higher grade achieved in the course.

The second part of the exam consists of 4 tasks (2 from each part of the course learning outcomes). Parts of the tasks need to be solved either on papar or using Excel. On each sheet of paper with the solutions you must write your name, surname and JMBAG as well as numerate the sheets and they need to be scanned after the exam finish. The PDF / Photo and Excel files need to be saved and uploaded onto Merlin or sent via e-mail. For this process you will have an additional 10 minutes.

Intended learning outcome 1:	Theory:	_____
	Exercises:	_____
Intended learning outcome 2:	Theory:	_____
	Exercises:	_____
T O T A L P O I N T S :		_____

EXERCISE 1

Exercise 1

- The small factory produces two types of screws V1 and V2. For 1 kg of V1 it is necessary to work on machine S1 for 2 h, and for 1 kg of V2 it is necessary to work on machine S1 for 1 h, and on machine S2 for 4 h. The capacities of the machines are limited: machine S1 10 h, and machine S2 12 h. What quantity of screws needs to be produced in order to maximize the profit, if HRK 20 is obtained for a kilogram of V1 and HRK 30 for a V2, provided that at least 2 kg of V1 is placed on the market?
- Mathematically formulate the problem of linear programming by setting the objective function, taking into account the set limits.
- Write the general form of the model and interpret the meaning of the structural and slack variables.
- Graphically solve the problem and find the optimal solution.
- Interpret the solution obtained by answering the following questions:
  - What are the optimal quantities of screws produced?
  - How much income was generated?
  - What is the situation with the model limitations?

Exercise 1

- Mathematically formulate the problem of linear programming by setting the objective function, taking into account the set limits.

	V1	V2	Constraints:
Machine S1	2	1	10 h
Machine S2	0	4	12 h
Min. Q of V1	1	0	2 kg
Price	20 kn	30 kn	← MAX!

GENERAL FORM:

$$MaxZ = 20x_1 + 30x_2$$

$$2x_1 + x_2 \leq 10$$
$$4x_2 \leq 12$$
$$x_1 \geq 2$$

$$x_1, x_2 \geq 0$$

Exercise 1

- Write the general form of the model and interpret the meaning of the structural and slack variables.

STANDARD FORM:

$$MaxZ = 20x_1 + 30x_2 + 0x_3 + 0x_4 + 0x_5$$

$$2x_1 + x_2 + x_3 = 10$$
$$4x_2 + x_4 = 12$$
$$x_2 - x_5 = 2$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Interpretation of variables:

STRUCTURAL:

$\boldsymbol{x_1}$: ammount of screws V1 in kg
$\boldsymbol{x_2}$: ammount of screws V2 in kg

SLACK:

$\boldsymbol{x_3}$: unused working hours of machine S1
$\boldsymbol{x_4}$: unused working hours of machine S2
$\boldsymbol{x_5}$: overslow over min. required ammount of V1 screws in kg

Exercise 1 ISPRAVI

- Graphically solve the problem and find the optimal solution.

1st constraint:

p1: $2x_1 + x_2 \leq 10$

$x_1 = 0$

$x_2 = 10$

[0; 10]

$x_2 = 0$

$2x_1 = 10 \quad /:2$

$x_1 = 5$

[5; 0]

2nd constraint:

p2: $4x_2 \leq 12$

$x_2 = 3$

[0; 3]

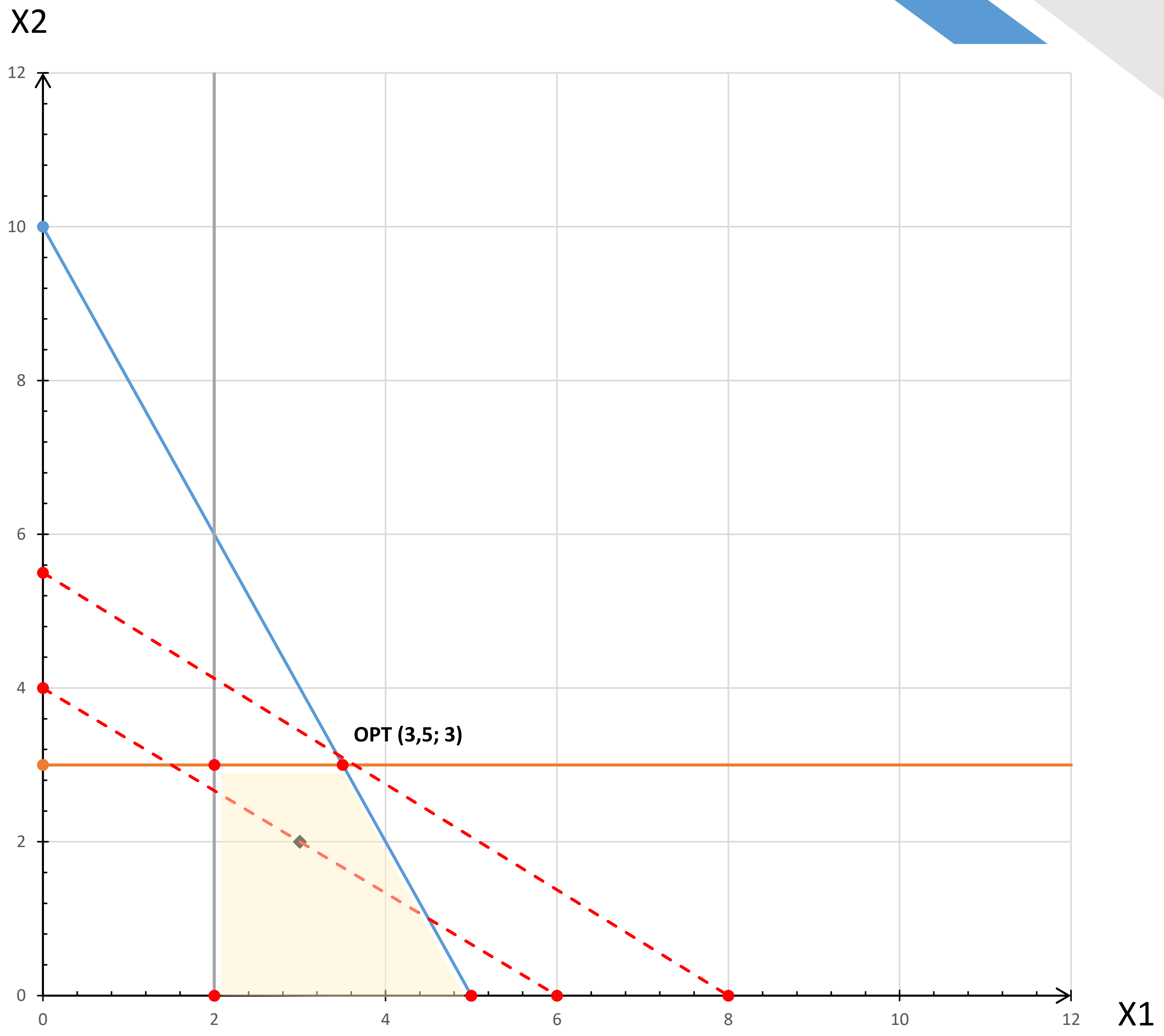
3rd constraint:

p3: $x_1 \geq 2$

$x_1 = 2$

[2; 0]

Exercise 1



T [3; 2]

$MaxZ = 20x_1 + 30x_2$

$20 \times 3 + 30 \times 2 = 120$

$20x_1 + 30x_2 = 120$

$x_1 = 0, \quad x_2 = 4$ [0; 4]

$x_2 = 0, \quad x_1 = 6$ [6; 0]

Exercise 1

- Interpret the solution obtained by answering the following questions:
  - What are the optimal quantities of screws produced?

    OPT [3,5; 3]

    The optimal amounts of production are 3,5 kg of V1 screws and 3 kg of V2 screws.
  - How much income was generated?

    $$MaxZ = 20 \times 3{,}5 + 30 \times 3 = 160$$

    The maximum profit is 160 HRK.
  - What is the situation with the model limitations?

    $$2 \times 3{,}5 + 3 = 10 = 10$$

    $$4 \times 3 = 12 = 12$$

    $$3{,}5 > 2$$

    The available working hours of both machines are used up entirely.
    The minimum requirement of screws V1 production is surpassed by 1,5 kg.

EXERCISE 2

Exercise 2

- The company is planning an advertising campaign to attract new customers and wants to place a total of no more than 10 ads in three daily newspapers. Each ad in newspaper A costs $ 200 and will be read by 2,000 people. Each ad in newspaper B costs $ 100 and will be read by 500 people. Each newspaper C ad costs $ 100 and will be read by 1,500 people. The company wants the ads to be read by at least 16,000 people in total. Determine the number of ads in each newspaper which the company will place in order to minimize advertising costs, if it is a known fact that newspaper C cannot publish more than 4 advertisements.
- Formulate the linear programming problem mathematically. Solve the problem using Excel. (Exercises 10 – Solutions.xlsx)
- Based on the answer report and the sensitivity report, answer the following questions:

Exercise 2

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$B$9	OF Min. Costs	0	1400

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$12	Variables Newspaper A	0	5	Contin
$C$12	Variables Newspaper B	0	0	Contin
$D$12	Variables Newspaper C	0	4	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$B$16	Max. Advertisements LS	9	$B$16<=$D$16	Not Binding	1
$B$17	Min. Readers LS	16000	$B$17>=$D$17	Binding	0
$B$18	Max. Ads in n. C LS	4	$B$18<=$D$18	Binding	0

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$12	Variables Newspaper A	5	0	200	200	66,66666667
$C$12	Variables Newspaper B	0	50	100	1E+30	50
$D$12	Variables Newspaper C	4	0	100	50	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$B$16	Max. Advertisements LS	9	0	10	1E+30	1
$B$17	Min. Readers LS	16000	0,1	16000	2000	10000
$B$18	Max. Ads in n. C LS	4	-50	4	4	4

Exercise 2

- Based on the answer report and the sensitivity report, answer the following questions:
  - How much does it cost to advertise the company?

    The minimum cost is $ 1,400.
  - In which newspapers did the company decide to place its advertisements and how many?

    The company paid for the publication of 5 advertisements in newspaper A and 4 advertisements in list B.
  - Are all restrictions met? Is there an overflow or unused resources in the limitations?

    A total of 9 advertisements were paid, 1 less than the maximum possible number.
  - Are there opportunity costs? If so, what are they saying?

    Yes, we did not decide to place advertisements in newspaper B. The opportunity cost is $ 150.

Exercise 2

- Based on the answer report and the sensitivity report, answer the following questions:
  - What is the price level of the advertisement in newspaper A, so that the company still decides to place 5 advertisements in that newspaper and 4 advertisements in newspaper C?

  The price of an advertisement can range from \$ 143.33 to \$ 400 (200 - 66.67 = \$ 143.33; 200 + 200 = \$ 400).

  - How will an allowable increase in the number of ads affect the overall cost of advertising?

  An increase in the possible number of ads will not affect the amount of the minimum cost (dual price: 0; allowable increase: infinite). With the current budget, it is possible to pay for 9 advertisements.

  - If the desired minimum number of people who will see an ad increases by 1,000, what impact will this have on the total cost of advertising?

  The minimum advertising cost will increase by \$ 10 (1,000 * 0.1 = \$ 10).

  - If newspaper C allowed more advertisements, how would that affect the company's costs?

  Up to the number of 8 advertisements in newspaper C with each additionally published advertisement in that list, the total costs would be reduced by \$ 50.

EXERCISE 3

Exercise 3

- An insurance company gave a list of ten of their customers, their age (in years) and their annual life insurance fee (in HRK). The data is given in the flowing table:

Customer	Cost of Insurance	Age
A	759	52
B	424	28
C	357	42
D	616	51
E	655	42
F	559	44
G	651	46
H	519	49
I	358	51

- Estimate the linear regression of insurance costs depending on the age of the customer.
- Write the evaluated regression function, interpret the meaning of the rated parameter and determine if the economic criterion is met in the model.
- If the age of a customer is 35 years, what will be the expected annual cost of insurance?

Exercise 3

$$\widehat{\beta_0} \times n + \widehat{\beta_1} \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} Y_i$$
$$\widehat{\beta_0} \sum_{i=1}^{n} X_i + \widehat{\beta_1} \sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n} X_i Y_i$$

- Estimate the linear regression of insurance costs depending on the age of the customer.

n = 9

Customer	Cost of Insurance (Yi)	Age (Xi)	Xi^2	Xi*Yi
A	759	52	2704	39468
B	424	28	784	11872
C	357	42	1764	14994
D	616	51	2601	31416
E	655	42	1764	27510
F	559	44	1936	24596
G	651	46	2116	29946
H	519	49	2401	25431
I	358	51	2601	18258
SUM:	**4898**	**405**	**18671**	**223491**

$$9\,\widehat{\beta_0} + 405\,\widehat{\beta_1} = 4898 \quad / \times (-45)$$
$$405\,\widehat{\beta_0} + 18671\,\widehat{\beta_1} = 223491$$

$$-405\,\widehat{\beta_0} - 18225\,\widehat{\beta_1} = -220410$$
$$405\,\widehat{\beta_0} + 18671\,\widehat{\beta_1} = 223491$$

$$446\,\widehat{\beta_1} = 3081 \quad / : 446$$
$$\widehat{\beta_1} = 6{,}9081$$

$$9\,\widehat{\beta_0} + 405 \times 6{,}9081 = 4898$$

$$\widehat{\beta_0} = 233{,}3577$$

Exercise 3

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \begin{bmatrix} n & X_i \\ X_i & X_i^2 \end{bmatrix}^{-1} \times \begin{bmatrix} Y_i \\ X_iY_i \end{bmatrix}$$

- Estimate the linear regression of insurance costs depending on the age of the customer.

n = 9

Customer	Cost of Insurance (Yi)	Age (Xi)	Xi^2	Xi*Yi
A	759	52	2704	39468
B	424	28	784	11872
C	357	42	1764	14994
D	616	51	2601	31416
E	655	42	1764	27510
F	559	44	1936	24596
G	651	46	2116	29946
H	519	49	2401	25431
I	358	51	2601	18258
SUM:	**4898**	**405**	**18671**	**223491**

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \begin{bmatrix} 9 & 405 \\ 405 & 18671 \end{bmatrix}^{-1} \times \begin{bmatrix} 4898 \\ 223491 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 405 \\ 405 & 18671 \end{bmatrix}^{-1} = \frac{1}{|detA|} \times A^* = \frac{1}{|9 \times 18671 - 405 \times 405|} \times \begin{bmatrix} 18671 & -405 \\ -405 & 9 \end{bmatrix}$$

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \frac{1}{|4014|} \times \begin{bmatrix} 18671 & -405 \\ -405 & 9 \end{bmatrix} \times \begin{bmatrix} 4898 \\ 223491 \end{bmatrix}$$

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \frac{1}{|4014|} \times \begin{bmatrix} 18671 \times 4898 - 405 \times 223491 \\ -405 \times 4898 + 9 \times 223491 \end{bmatrix}$$

$$\begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \frac{1}{|4014|} \times \begin{bmatrix} 936703 \\ 27729 \end{bmatrix} \qquad \begin{bmatrix}\widehat{\beta_0}\\ \widehat{\beta_1}\end{bmatrix} = \begin{bmatrix} 233,3590 \\ 6,9081 \end{bmatrix}$$

Exercise 3

- Write the evaluated regression function, interpret the meaning of the rated parameter and determine if the economic criterion is met in the model.

$$\widehat{Yi} = 233{,}3590 + 6{,}9081\ Xi$$

With the increase of customer age by 1 year, the cost of insurance will increase by 6,9081 HRK.

- If the age of a customer is 35 years, what will be the expected annual cost of insurance?
- $Xi = 35$

$$\widehat{Yi} = 233{,}3590 + 6{,}9081 \times 35$$
$$\widehat{Yi} = 457{,}14$$

With the age of 35, a customer will need to pay 457,14 HRK insurance cost annually.

EXERCISE 4

Exercises 4

- Data were collected for a sample of 104 weekly sales volumes of pet food and variables that affect the amount of those values (Exercises 10.):

  Y: sales volume (kg / week),
  X1: average prices (USD / kg),
  X2: log sales volume (%),
  X3: log average price (%),
  X4: discount price: 1 = Yes, 0 = No.

- Enter the correct dummy variables for the data.
- Determine the equation of the multiple linear regression model. Explain the meaning of the parameters.
- Assess the statistical significance of the model.
- Estimate the statistical significance of the regression coefficients (regression variables) with significance level 1%.
- Test the statistical significance of the regression model with significance level of 5%.

Exercise 4

SUMMARY OUTPUT

Regression Statistics	
Multiple R	0,997934514
R Square	0,995873293
Adjusted R Square	0,995706558
Standard Error	1100,698621
Observations	104

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	4	28944932024	7236233006	5972,768716	4,7196E-117
Residual	99	119942208	1211537,454		
Total	103	29064874232			

	Coefficients	*Standard Error*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	-829016,8163	21466,1846	-38,61966306	1,67886E-61	-871610,3837	-786423,249
Avg Price ($)	408091,5265	9715,568335	42,00387589	6,59613E-65	388813,7312	427369,3219
Log Sales Volume	43852,94344	1831,568773	23,942832	5,72469E-43	40218,71363	47487,17324
Log Avg Price	-472478,8684	11845,75199	-39,88593284	8,33388E-63	-495983,4103	-448974,3265
Dummy Discount Prices	-117,2781878	225,364881	-0,520392473	0,603951913	-564,4510051	329,8946294

Exercises 4

- Determine the equation of the multiple linear regression model. Explain the meaning of the parameters.

$$\hat{Y}_i = -829016{,}8163 + 408091{,}5265X_{1i} + 43852{,}9434\, lnX_{2i} - 472478{,}8684\, lnX_{3i} - 117{,}2782X_{4i}$$

- Interpretation:

If the average prices increase by 1 $, it is expected that the sales volume will increase by 408.091,52 $, cet. par.

If the sales would increase by 1 %, it is expected that the sales volume will increase by 438,5294 $ (43852,9434/100 units), cet. par.

If the average prices would increase by 1 %, it is expected that the sales volume will decrease by 4.724,7887 $ (472478,8684/100 units), cet. par.

If the pet food were on discount, it is expected that the sales volume will be smaller by 117,28 $, cet. par.

Exercises 4

- Assess the statistical significance of the model.

  $R^2 = 0{,}9958$ 99.58 % of the variance of the dependent variable was explained by the model.

  $\overline{R^2} = 0{,}9957$

  $s = 1100{,}6986$

  The standard error of the model is $ 1.100,6986.

  (Take the average value of the dependent variables (function: =AVERAGE(range)) and divide the standard error value with the average. → It will give you the standard error value as a percentage. → Under 10 % is low level.)

  The model fit to reality is high. The coefficients of determination are high and the standard error of the model is low.

Exercises 4

- Estimate the statistical significance of the regression coefficients (regression variables) with significance ~~level~~ T-TEST! 1%.

$\alpha = 0{,}01$

$n = 104$

$k = 4$

$df = n - k - 1 = 99$

$t_c = 2{,}626$

1. Hypothesis:

$H_0: \quad \beta_1 = 0$

$H_A: \quad \beta_1 \neq 0$

$H_0: \quad \beta_2 = 0$

$H_A: \quad \beta_2 \neq 0$

2. Testing:

$t_{\widehat{\beta_1}} = 42{,}0039$

$|t_1| > t_c$

$|42{,}0039| > 2{,}626$

$\rightarrow H_A!$

$t_{\widehat{\beta_2}} = 23{,}9428$

$|t_{\beta_2}| > t_c$

$|23{,}9428| > 2{,}626$

$\rightarrow H_A!$

3. Conclusion:

With a significance level of 1 %, we accept hypothesis Ha and conclude that average prices in $ have a statistically significant effect on the sales volume of pet food.

With a significance level of 1 %, we accept hypothesis Ha and conclude that sales volume change in % have a statistically significant effect on the sales volume of pet food.

Exercises 4

- Estimate the statistical significance of the regression coefficients (regression variables) with significance T-TEST! ~~level~~ 1%.

$\alpha = 0{,}01$

$n = 104$

$k = 4$

$df = n - k - 1 = 99$

$t_c = 2{,}626$

1. Hypothesis:

H_0: $\beta_3 = 0$ $\qquad$ H_0: $\beta_4 = 0$

H_A: $\beta_3 \neq 0$ $\qquad$ H_A: $\beta_4 \neq 0$

2. Testing:

$t_{\widehat{\beta_3}} = -39{,}8859$ $\qquad$ $t_{\widehat{\beta_4}} = -0{,}5204$

$|t_{\beta_3}| > t_c$ $\qquad$ $|t_{\beta_4}| < t_c$

$|-39{,}8859| > 2{,}626$ $\qquad$ $|-0{,}5204| < 2{,}626$

→ H_A! $\qquad$ → H_0!

3. Conclusion:

With a significance level of 1 %, we accept hypothesis Ha and conclude that average prices change in % have a statistically significant effect on the sales volume of pet food.

With a significance level of 1 %, we accept hypothesis Ha and conclude that discnout prices have not a statistically significant effect on the sales volume of pet food.

Exercises 4

- Test the statistical significance of the regression model with significance level of 5%.

F-TEST:

$\alpha = 0{,}05$

$n = 104$

$k = 4$

$df = n - k - 1 = 99$

$F_c = 2{,}464$

1. Hypothesis:

H_0: $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$

H_A: H_o $is\ false$

2. Testing:

$F = 5.972{,}7687$

$F > F_c$

$5.972{,}7687 > 2{,}464$

→ H_A!

Interpretation:

With a significance level of 5 %, we accept hypothesis Ha and conclude that the model is statistically significant.

Thank you!

Questions?