Mathematics for economic and financial analysis - exercise classes

Prudky, Ivan

Mathematics for economic and financial analysis

Exercise classes – intended learning outcome 2

Academic year 2021/2022

Teaching assistant: Ivan Prudky, ivan.prudky@gmail.com

Ivan Prudky, ivan.prudky@gmail.com

Simple interest rate

Interest account

- *INTEREST - fee paid by the debtor for the borrowed amount (principal) for a certain period of time*
- *INTEREST RATE - the amount of interest on a certain amount of monetary units for a certain time interval*
- *CAPITALIZATION PERIOD - time interval of "renting" money [year, semester, quarter, month…]*

Calculation of interest

- *the interest rate is prescribed by law or contract between the creditor and the debtor*
- *ANTICIPATIVE - at the beginning of the accounting period in relation to the principal at the end of the period*
- *DECURSIVE - at the end of the accounting period in relation to the principal from the beginning of the period*
- **EXAMPLE: We borrowed HRK 1,000.00 per month with an interest rate of 10%.*

	ANTICIPATIVE	**DECURSIVE**
BORROWED	900,00 HRK	1.000,00 HRK
RETURNED	1.000,00 HRK	1.100,00 HRK

Types of interest

- *SIMPLE - interest is always calculated on the same principal value; interest rates are always the same*
- *APPLICATION: securities (checks, bills of exchange), consumer loans*
- *COMPOUND - the principal for the calculation of interest changes for each period; interest rates change*
- *APPLICATION: periodic payments and withdrawals, investment loans*

Simple vs. compound interest

- *We invested HRK 10,000.00 for 3 years with 5% decursive annual interest.*

TIME	**SIMPLE**	**COMPOUND**
1.	10.000,00 + 500,00	10.000,00 + 500,00
2.	10.500,00 + 500,00	10.500,00 + 525,00
3.	11.000,00 + 500,00	11.025,00 + 551,25
	11.500,00	11.576,25

Ivan Prudky, ivan.prudky@gmail.com

Simple interest account

Interest:
$$I = \frac{C \times p(G) \times n}{100}$$

Interest rate:
$$p(G) = \frac{I \times 100}{P \times n}$$

Time:
$$n = \frac{I \times 100}{P \times p(G)}$$

Principal:
$$P = \frac{I \times 100}{n \times p(G)}$$

Future value of the capital:
$$P_n = P + I$$
$$P_n = P + \frac{P \times p(G)}{100} \times n$$
$$P_n = P\left(1 + \frac{p(G) \times n}{100}\right)$$

Notation:	Meaning:
P	Capital/principal
I	Simple interest
$p(G)$	Annual interest rate
n	Time (years)
P_n	Future value of capital

Simple interest account

1) How much is the simple interest on the capital of HRK 50.000 for a period of three years and with an annual interest rate of 8? The calculation of interest is decursive.

2) After one year and six months, the debtor repaid the borrowed amount of HRK 6.000 along with a simple interest value of HRK 801. What was the annual interest rate? The calculation of interest is decursive.

3) With which annual simple interest rate and decursive interest calculation does an amount increase by 60% in 9 years? The calculation of interest is simple and decursive.

4) With which annual interest rate does an amount of capital quadruple its value in 10 years? The calculation of interest is simple and decursive.

Ivan Prudky, ivan.prudky@gmail.com

Basic scales

- *when time is expressed in days and months*
- *a specific time period of interest*

Method	***Year***	***Month***
German	*360 days*	*30 days*
French	*360 days*	*calendar*
English	*365 days*	*calendar*

- *Standard in HR: English method*
- *366 days in case of leap year*
- *If the method is not stated, we use English*
- *If the year is not stated, it is not a leap year*
- *The first day is not taken into account*
- *The last day is taken into account*

n = days $n = \frac{d}{365}$ $d = \frac{I \times 36500}{P \times p(G)}$ $I = \frac{P \times p(G) \times d}{36500}$

n = months $n = \frac{m}{12}$ $m = \frac{I \times 1200}{P \times p(G)}$ $I = \frac{P \times p(G) \times m}{1200}$

1) In how many days does the amount of HRK 85.000, with an annual interest rate of 6, bring HRK 12.750 of simple interest? The calculation of interest is decursive.

2) The company raised a short-term working capital loan of HRK 500.000 with 8% annual interest for the period from 13.01. to 16.09. the same year. The calculation of interest is decursive. How much are the simple interests according to:
   a. English method;
   b. German method;
   c. French method?

3) The company raised a short-term working capital loan of HRK 500.000 with 8% annual interest for the period from 13.01. to 16.09. the same year. The calculation of interest is decursive. How much are the simple interests according to:
   a. English method;
   b. German method;
   c. French method?

4) The company raised a short-term working capital loan of HRK 500.000 with 8% annual interest for the period from 13.01. to 16.09. the same year. The calculation of interest is decursive. How much are the simple interests according to:
   a. English method;
   b. German method;
   c. French method?

Ivan Prudky, ivan.prudky@gmail.com

Interest bill higher and lower one hundred

- *it is used when the known principal is increased/decreased by the amount of interest*
- *necessary to determine: value of principal, interest…*

Annual calculation:
$$P = \frac{(P \pm I) \times 100}{100 \pm p(G) \times n} \qquad I = \frac{(P \pm I) \times p(G) \times n}{100 \pm p(G) \times n}$$

Monthly calculation:
$$P = \frac{(P \pm I) \times 1200}{1200 \pm p(G) \times m} \qquad I = \frac{(P \pm I) \times p(G) \times m}{1200 \pm p(G) \times m}$$

Daily calculation:
$$P = \frac{(P \pm I) \times 36500}{36500 \pm p(G) \times d} \qquad I = \frac{(P \pm I) \times p(G) \times d}{36500 \pm p(G) \times d}$$

1) The loan of HRK 10.000,00, approved on 15 May, was repaid by the debtor together with the related simple interest of 7% per annum, which totalled HRK 10.750,00. On what date is the debt settled if the interest is decursive?

2) After deducting 5% interest on the loan from 01.03 to 30.04, the bank paid the debtor the remaining HRK 50.450,00. What was the simple interest rate if the calculation was annual and decursive?

3) How much interest will Marija receive for 2021 if the following information is written in the savings book:

DATE	WITHDRAWAL	PAYMENT	BALANCE
10.01.2021		3.000,00	3.000,00
23.01.2021	1.250,00		1.750,00
08.02.2021		4.500,00	6.250,00
16.02.2021	1.750,00		4.500,00
05.03.2021		1.000,00	5.500,00
17.03.2021	500,00		5.000,00

The interest rate is 5.

- *1st method of interest calculation:*
  - *determine the number of days for each payment and each withdrawal from the date of payment/withdrawal to the end of the billing period*
  - *calculate all interest payments*
  - *calculate all interest withdrawals*
  - *difference between interest on payments and withdrawals*
- *2nd method of interest calculation:*
  - *for each balance, the days from that date to the following change are calculated*
  - *simple interest is calculated for each balance → the sum of them is the final value*

4) What amount of interest will Marko receive for 2021 if the following data is written on the savings book, with an interest rate of 4%:

DATE	PAYMENT	WITHDRAWAL	BALANCE
28.01.2021		6.000,00	6.000,00
04.02.2021	1.500,00		4.500,00

Ivan Prudky, ivan.prudky@gmail.com

13.03.2021		3.500,00	8.000,00
22.03.2021	2.500,00		5.500,00
26.04.2021		8.000,00	13.500,00
06.05.2021		5.000,00	18.500,00
14.06.2021	4.250,00		14.250,00
30.06.2021		1.350,00	15.600,00
23.07.2021	2.225,00		13.375,00
16.08.2021		6.000,00	19.375,00

Bill of exchange account

- *BILL OF EXCHANGE = a security that contains a person's obligation to accurately pay a certain amount of money to another person on a specific date*
- *Own bill of exchange → the issuer itself has an obligation*
- *Traced bill of exchange → the issuer imposes an obligation on a third-party*
- *they can be redeemed or sold:*

1. *on the date indicated on the bill of exchange;*
2. *before the due date of the bill;*
3. *after the due date of the bill of exchange.*

- *upon sale/purchase, the bank earns: commission of discount value + transaction costs of the bill of exchange*

1) The bill of exchange reads HRK 150.000, and the due date is June 24. What is its value if the annual interest rate is 6:
   a. May 14 of the same year;
   b. July 1 of the same year?

2) Calculate the sale of a bill of exchange in the amount of HRK 50.000,00 payable on March 25, which the company sold to the bank on February 13 with a 7,5% discount, 3 per mille of commission and HRK 43,50 of costs.

3) Three bills of exchange read:
   HRK 22.000,00 with due date 3.3. and interest of 4,5% per annum,
   HRK 26.000,00 with due date 4.4. and interest of 5,25% per annum,
   HRK 30.000,00 with due date 5.5. and interest of 7,5% per annum.
   Bills of exchange are replaced by a new one equal to the sum of their nominal amounts. The due date of the new bill of exchange needs to be determined.

4) The four bills of exchange read as follows:
   HRK 5,000.00 with a due date of January 2;
   HRK 6,000.00 with a due date of February 3;
   HRK 7,000.00 with a due date of March 4;
   HRK 8,000.00 with a due date of April 5,
   which needs to be replaced with a new bill of exchange due on March 19. What will be the nominal amount on the new bill of exchange if the annual discount is 10?

5) Calculate the sale of a bill of exchange in the amount of HRK 35.750,00 payable on April 26, which the company sold to the bank on February 9 with a 3,75% discount, 4,5 per mille of commission and HRK 50,00 of costs.

6) Three bills of exchange read on:
HRK 12,250.00 with a due date of February 4;
HRK 13,750.00 with a due date of May 6;
HRK 15,000.00 with a due date of June 14;
which needs to be replaced with a new bill of exchange owing on April 26, 2020. What will be the nominal amount on the new bill of exchange if the annual discount is 5?

7) Calculate the sale of a bill of exchange in the amount of HRK 35.750,00 payable on April 26, which the company sold to the bank on February 9 with a 3,75% discount, 4,5 per mille of commission and HRK 50,00 of costs.
P = 35.750 kn
p(G) = 3,75
d = 20 + 31 + 26 = 77

I → interests
I = ? $I = (35.750 \times 3{,}75 \times 77)/36.600 = 282{,}04$

S → discounted bill of exchange value
S = ? $S = 35.750 - 282{,}04 = 35.467{,}96$

Pr → commission in per mille
Pr = ? $Pr = (S \times p(S))/1.000 = (35.467{,}96 \times 4{,}5)/1.000 = 159{,}61$

a) Nominal amount of bill of exchange (P)	**35.750,00**
b) Interest amount (discount 3,75 %) (I)	282,04
c) Discounted value of the bill of exchange (S) (a - b)	35.467,96
d) Commission amount (4.5 per thousand to discounted value) (Pr)	159,61
e) Accrued expenses (banking)	50,00
f) Bill of exchange value (c - d - e)	**35.258,35**

8) Three bills of exchange read on:
HRK 12,250.00 with a due date of February 4;
HRK 13,750.00 with a due date of May 6;
HRK 15,000.00 with a due date of June 14;
which needs to be replaced with a new bill of exchange due on April 26, 2020. What will be the nominal amount on the new bill of exchange if the annual discount is 5?
p(G) = 5
d1 = 82 days d2 = 10 days d3 = 49 days

$$fiveX = 12.250 + (12.250 \times 5 \times 82)/36.600 + 13.750 - (13.750 \times 5 \times 10)/36.600$$
$$+ 15.000 - (15.000 \times 5 \times 49)/36.600 =$$
$$X = 12.387{,}23 + 13.731{,}22 + 14.899{,}59 = 41.018{,}04n$$

Ivan Prudky, ivan.prudky@gmail.com

Compound interest rate

- *Decursive calculation of interest*
  - *Final/future value of principal*
  - *Variable decursive interest rate*
  - *Initial/present value of principal*
  - *Relative interest rate*
  - *Conformal interest rate*
- *Anticipatory calculation of interest*
  - *Final/future value of principal*
  - *Initial/present value of principal*
  - *Nominal, relative and conformal interest rates*
- *Equivalent decursive/anticipative interest rate*

The final value of the principal

- *Compound interest account → for periods longer than one year*
- *Decursive calculation of interest: interest is added to the principal (variable principal)*

Notation:

P → original principal, present value
I → the amount of compound interest
P_n → compound amount of P, accumulated value of P
n → total number of interest periods involved
m → number of interest periods per year, frequency of compounding
j_m → nominal (yearly) interest rate, which is compounded
i → interest rate per interest period

The final value of the principal

Final value at the end of the nth year: $P_n = P \times (1 + {}^{j}/_{100})^n$ → *final value*

$P_n = P \times i^n$ → *final value*

$i = \sqrt[n]{{}^{P_n}/_{P_0}}$ → *interest rate factor*

$n = \frac{\log {}^{S_n}/_{S_0}}{\log i}$ → *interest time*

- *reality: the decursive annual interest rate is variable over time*

$i \; = 1 + \frac{j}{100}$ → *variable decursive interest rate factor*

$P_n = P \times i^n$ → *final value*

$i = \sqrt[m]{i_1 \times i_2 \times \ldots \times i_m}$

Ivan Prudky, ivan.prudky@gmail.com

$j = 100(i - 1)$ → *average interest rate*

1) The amount of HRK 10,000 was invested for 5 years with a compound, annual decursive interest rate of 4.5%. Calculate the final principal amount.

2) One person invested HRK 80,000 in the bank today. The bank approves 7.5% annual interest. How much will he have at the end of the 6th year if the calculation of interest is compound, annual and decursive?

3) Today, the investor deposited HRK 800,000 in the bank. In this case, the bank approved 10% annual interest. How much will he have at the end of the fifth year if the interest calculation is decursive, annual, and compound?

4) When does a principal increase with compound interest to 250% if interest is calculated at an annual interest rate of 7.5? The calculation of interest is decursive, annual and compound.

5) Today, a person invested HRK 40,000 in the bank. How much will he have at the end of the sixth year if the bank approves an interest rate of 7.5% for the first three years and 10% annual interest for the last three years? Interest is calculated decursively, annually, and compounded.

6) What amount invested in the bank today will increase in 10 years, together with compound interest, to HRK 65,000? The bank calculates 7.5% of annual interest rates on a recursive basis.

7) What amount invested today in the bank brings 90,000 kn with compound interest in 8 years? The bank charges 6% for the first two years and 7% for the remaining period of annual interest rates on a decursive basis.

8) How many HRK were invested 10 years ago, if today there are 30,000 HRK left in the savings account, and 20,000 HRK were raised 5 years ago? In addition to compound, annual and decursive calculations, the bank calculated:
   a. throughout the time, 7.5% annual interest.
   b. the first 5 years 7.5%, and the last 5 years 8% annual interest.

9) How many HRK were invested 10 years ago, if today there are 30,000 HRK left in the savings account, and 20,000 HRK were raised 5 years ago? In addition to compound, annual and decursive calculations, the bank calculated:
   a. throughout the time, 7.5% annual interest.
   b. the first 5 years 7.5%, and the last 5 years 8% annual interest.

Ivan Prudky, ivan.prudky@gmail.com

Interest rates

- *Relative*

$$j_r = {}^{j}/_{m}$$

- *Conformal*

$$j' = 100 \times \left[\left(1 + {}^{j}/_{100}\right)^{\frac{1}{m}} - 1 \right]$$

- *m – interest calculation number*

$$m = \frac{n1}{n2}$$

  - *n1 = the length of the time interval for which the nominal interest rate is prescribed*
  - *n2 = the length of the time interval in which interest is charged/calculated*

1) Determine the final value of the deposit of HRK 50,000.00 after 8 years with a nominal annual interest rate of 12% if the interest is paid:
   a. annually,
   b. semi-annually with the application of the relative interest rate,
   c. bimonthly with the application of relative interest rate.

2) Determine the final value of the deposit of HRK 50,000.00 after 8 years with a nominal annual interest rate of 12% if the interest is paid:
   a. annually,
   b. semi-annually with the application of the relative interest rate,
   c. bimonthly with the application of relative interest rate.

3) Determine the final value of the deposit of HRK 50,000.00 after 8 years with a nominal annual interest rate of 12% if the interest is paid:
   a. annually,
   b. semi-annually with the application of the relative interest rate,
   c. bimonthly with the application of relative interest rate.

4) Determine the final value of the deposit of HRK 20,000.00 after 4 years with a default monthly interest rate of 1.5% if the capitalisation is:
   a. monthly,
   b. bimonthly,
   c. semi-annual.

5) Determine the final value of the deposit of HRK 20,000.00 after 4 years with a default monthly interest rate of 1.5% if the capitalisation is:
   a. monthly,
   b. bimonthly,
   c. semi-annual.

6) Determine the final value of the deposit of HRK 20,000.00 after 4 years with a default monthly interest rate of 1.5% if the capitalisation is:

a. monthly,
b. bimonthly,
c. semi-annual.

7) Determine the value to which the amount of HRK 50,000.00 is invested in the bank with 2% semi-annual interest increases after 6 years.
Calculate the comfortable monthly interest rate and check whether we get the same value from the monthly interest accrual.
What would be the appropriate interest rate that would give the same amount of interest?

8) Determine the value to which the amount of HRK 50,000.00 is invested in the bank with 2% semi-annual interest increases after 6 years.
Calculate the comfortable monthly interest rate and check whether we get the same value from the monthly interest accrual.
What would be the appropriate interest rate that would give the same amount of interest?

9) Determine the value to which the amount of HRK 50,000.00 is invested in the bank with 2% semi-annual interest increases after 6 years.
Calculate the comfortable monthly interest rate and check whether we get the same value from the monthly interest accrual.
What would be the appropriate interest rate that would give the same amount of interest?

10) Today, a person deposited HRK 80,000.00 in the bank. How much will it have at the end of the 6th year if the bank approves 6% annual interest in the first two years and 8% in the remaining years? The calculation of interest is compound, semi-annual and decursive.

11) One year ago, HRK 45,000.00 was invested in the bank. How much money should additionally be paid today to raise HRK 500,000.00 after 5 years? The calculation of interest is semi-annual and decursive, and the bank approves the annual interest rate 5 (use the relative interest rate).

Final values of multiple periodic payments/withdrawals

Introduction:

- *Consider the case where several equal amounts are paid (paid out) evenly at equal time intervals over n periods. Assume that the capitalisation period is equal to the maturity period between these payments and that the interest rate is constant*
- *Payments can be at the beginning of the period, so we talk about PRENUMERANDO PAYMENTS (PAYMENTS) or at the end of the period, so we talk about POSTNUMERANDO PAYMENTS (PAYMENTS)*
- *We want to calculate the final value of all these payments (payments), i.e. replace all these equal payments R with one sum at the end of the nth period*

Prenumerando

- *Let the interest calculation be decursive, and the payments will be made at the beginning of the period. To calculate the final value of payments should be calculated as follows:*

- *at the beginning of the first period is the first payment: R*
- *at the beginning of the second period, the second payment R follows and the first amount accrued for one period is added to it:* $R + R \times i$
- *at the beginning of the third period comes the third payment R, and to it is added the second interest for one period and the first interest now for two periods:* $R + R \times i + R \times i^2$
- *In the end, equality is obtained:* $S_n = R \times i \times \frac{i^n - 1}{i - 1}$

Postnumerando

- *The final value of postnumerando payments is calculated according to the same procedure as those of prenumerando. The DIFFERENCE is that payments start one period later and that the last payment should not be accrued because it is due at the end of the nth period.*
- *In the end, equality is obtained:* $S_n' = R \times \frac{i^n - 1}{i - 1}$

1) At the beginning of each year, a person invests HRK 15,000 in a bank for 5 years. If the annual interest rate is 7%, what is the final value of all payments at the end of 5 years, and what is the final value at the end of 8 years? Interest is calculated annually, compoundly, and decursively.

2) At the beginning of each year, a person pays 10,000 HRK at the beginning of each year at 10% of the annual interest rate, and in the next seven years, 15,000 HRK at 8% of the annual interest rate. How much will that person have in the bank at the end of the fifteenth year if the annual interest rate of 8% is still applied?

3) A person invests in a bank in the name of housing savings at the end of each month for HRK 500. How much will that person have in the bank at the end of the 10th year if the state incentive funds of HRK 1,250 are paid to him at that time? The monthly interest rate is 1.25%, and the interest calculation is monthly, compound and decursive.

4) At the end of the fifth year, a person must pay the amount of 30,000 HRK. In agreement with the creditor, the debtor undertook to pay a certain amount at the end of each year for five years. What is the amount if the creditor requests 8% annual interest, and the calculation is annual, compound and decursive?

5) HRK 100,000.00 was invested in the bank so that in the next 10 years, HRK 12,000.00 could be raised at the end of the year. The calculation of interest is compound, annual and decursive, with an annual interest rate of 10%. How much will be in the bank account at the end of the tenth year?

6) A person paid 5,000 euros to the bank at the beginning of each year for 5 years. How much will that person have in the bank at the end of the tenth year if the bank applies an interest rate of 6% for the first three years, 5% in the remaining period and if the person withdrew the amount of 10,000 euros in the seventh year?

Ivan Prudky, ivan.prudky@gmail.com

Initial values of multiple periodic payments/withdrawals

Question:

- *How much money do we have to have today in an account at a bank (or a similar financial institution) to allow ourselves some fixed amount (rent) every month (year, ...) for the next 5, 10, 25 years?*
- *We must calculate the present (initial) value of multiple periodic payments (withdrawals). Payments can be made at the end or start of the period to distinguish between the two cases.*

Postnumerando

- *We are looking for the initial value of all n number same-value payments (R) paid off at the end of each period with the interest rate of j. The interest calculation is complex and decursive, and the capitalisation period is equal to the time period between the maturity of those payments. The initial value An for n postnumerando payments is equal to:* $A_n = R \times \frac{i^n - 1}{i^n (i-1)}$

Prenumerando

In the case where payments (withdrawals) are at the beginning of the period, the initial value of all n prenumerando payments is equal to: $A'_n = R \times \frac{r^n - 1}{r^{n-1}(r-1)}$

1) What amount of money should be invested today to ensure five annual postnumerando payments per 30,000 HRK? The interest calculation is annual, complex, and decursive, with an annual interest rate of 5%.

2) Based on the amount of HRK 60,000 invested, what is the value of the annual withdrawals at the end of each year over the next 4 years? The interest calculation is annual, compound, and decursive, with an annual interest rate of 3.5%.

3) How many years will you be able to raise HRK 5,000 annually, at the end of each year, based on a one-off payment of HRK 40,000? The interest calculation is annual, compound, and decursive, with an annual interest rate of 3 %.

4) What amount should be invested today in the bank to provide six annual prenumerando payments of 10,000 kn? Interest calculation is annual, compound, and decursive, with an annual interest rate of 4.5%.

5) Based on the amount of HRK 145,000 invested, how many equal annual amounts can be withdrawn at the beginning of each year over the next 7 years? The interest calculation is annual, compound, and decursive, with an annual interest rate of 2.5%.

Ivan Prudky, ivan.prudky@gmail.com

Perpetuities

- *If we want the number of rent to be infinite, that is, if we want to receive the perpetual rent based on the sum we have, we need to calculate the marginal value An of when the number of periods tends to infinity.*
- *Suppose we want to provide countless postnumerando rent based on the sum of the amount of R. Let this sum be invested in the bank with compound decursive capitalisation and interest rate j (with the appropriate decursive interest factor r).*
- *Perpetuities*
- *If we want to ensure perpetual postnumerando and prenumerando rents R, with a decursive interest rate of j, we will calculate:*

$$A'_\infty = \frac{R}{r-1} \qquad\qquad A_\infty = \frac{R\times r}{r-1}$$

1) Calculate how much we have to invest at the beginning of next year if we want to receive a perpetual prenumerando rent of HRK 20,000 at the beginning of each year. The annual decursive interest rate is 10 %.

2) The principal of HRK 100,000 provides a postnumerando perpetual annual rent of HRK 12,000. Capitalisation is annual, compound, and decursive. At what annual interest rate were the savings approved?

3) Is it more favourable to sell a three-bedroom flat in Zagreb, which has a market value of € 1,200,000, with the obtained money deposited at a commercial bank account that pays a 6% annual interest (postnumerando) or rent the flat for an annual net rent of € 80,000?

4) Is it more favourable to sell a three-bedroom flat in Zagreb, which has a market value of € 1,200,000, with the obtained money deposited at a commercial bank account that pays a 6% annual interest (prenumerando) or rent the flat for an annual net rent of € 80,000?

5) At what level will the prenumerando interest rate on the deposited funds be more profitable to sell the flat from the previous task per market value of € 1,200,000 compared to € 80,000 annually net rental?

Continuous compounding interest

- *Continuous compounding interest = Interests are calculated and drawn to the principal "at all times."*
- *in practice, pointless*
- *BUT: natural growth rate, medicine, macroeconomic*
- *Interests are attributed to the principal every moment (continuous, continuously)*

$$P_n = P_0 \times i^n = P_0 \times \left(1 + \frac{j}{100}\right)^n$$

Ivan Prudky, ivan.prudky@gmail.com

- *If j is an annual interest rate and the interest calculation is done m times, we have:*

$$P_n = P_0 \times \left(1 + \frac{j/_m}{100}\right)^{n\times m}$$

$$\lim_{m\to\infty} P_n = \lim_{m\to\infty} P_0 \times \left(1 + \frac{j}{100m}\right)^{100m\times\frac{n}{100}} = \left\{\lim_{X\to\infty}\left(1+\frac{k}{X}\right) = e^k\right\} = C_0 \times e^{j\times\frac{n}{100}}$$

- *FORMULA FOR CONTINUOUS COMPOUNDING:*

$$P_n = P_0 \times e^{\frac{n\times j}{100}}$$

1) How many kg will a child have on their 6th birthday if born with 3.5 kg and if it is assumed that the first two years are continuously gaining 5% additional weight per month and the last 4 years have 2% a month?

2) In 1991, according to the official census of the Republic of Croatia, Croatia had a population of 4,784,265, and in 2001 the value was 4,437,460. If the annual growth rate remains unchanged, what is the expected number of inhabitants in 2011?

3) In 1991, according to the official census of the Republic of Croatia, Croatia had a population of 4,784,265, and in 2011 the value was 4,284,889. If the rate of annual growth rate remains unchanged, what is the expected number of inhabitants in 2021?

Po = 4.784.265	$P_n = P_0 \times e^{\frac{n\times j}{100}}$
Pn = 4.284.889	$4.284.889 = 4.784.265 \times e^{20p/100}$ / : 4.784.265
n = 10 god	$0{,}895621166 = e^{j/5}$ / *ln*
j = ?	$\ln 0{,}895621166 = \frac{j}{5}\ln e$ / $\times 5$
	$j = 5 \times \ln 0{,}895621166 = \mathbf{-0{,}5512}\ \%$
	$P_{2021} = P_{2011} \times e^{\frac{n\times j}{100}} = 4.284.889 \times e^{\frac{10\times(-0{,}5512)}{100}} = \mathbf{4.055.097{,}16}$

Loans with equal annuity (periodic payment)

Loan

- *ANNUITY is a periodic amount paid by the loan user and consists of two parts: a PAYMENT QUOTA (a part to repay the nominal loan amount) and INTEREST*
- *Basic prerequisites for the repayment model of equal annuities: interest calculation is compound and decursive, the annuities are equal and repaid in equal periods at the end of the term, the period of interest calculation is equal to the period of time between the annuities, the interest rate is constant*

Notation:

P = loan value
a = annuity
I_k = interest at the end of the kth period
Rk = payment quota at the end of the kth period
P_k = outstanding principal at the end of the kth period
j = constant interest rate

Loan

- *Loan P must be equal to the current value of n postumerando annuities:*

$$P = a \times \frac{i^n - 1}{i^n (i-1)} \quad \rightarrow \quad a = P \times \frac{i^n (i-1)}{i^n - 1}$$

- *Interests are obtained from the rest of the debt from the previous period:*

$$I_k = P_{k-1} \times \frac{j}{100}$$

- *Payment quota Rk is a difference between annuity and interest:*

$$R_k = a - I_k \quad \rightarrow \quad R_k = R_1 \times i^{k-1}$$

- *The outstanding principal is the difference of previous debt and current payment quota:*

$$P_k = P_{k-1} - R_k$$

REPAYMENT TABLE

k	a	Ik	Rk	Pk
0	$-$	$-$	$-$	$P0$
1	a	$I1$	$R1$	$P1$
2	a	$I2$	$R2$	$P2$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$n-1$	a	$In-1$	$Rn-1$	$Pn-1$
n	a	In	Rn	0
$\sum$	$n \times a$	$I = \sum_{k=1}^{n} I_k$	$P = \sum_{k=1}^{n} R_k$	

1) The loan of EUR 50,000.00 was obtained for 4 years of repayment with 8 % of the annual interest and payment of equal annuities at the end of each year. Make the repayment table.

2) The loan was approved to the company for 5 years with 6 % of the annual interest and is repaid in nominally equal annuities at the end of the year in the amount of EUR 50,000.00. Determine the value of the loan. The calculation of interest is compound, annual and decursive. Make a repayment plan.

n = 5 years $\quad P = a \times \frac{i^n - 1}{i^n(i-1)}$

j = 6 % → i = 1,06 $\quad P = 50.000 \times \frac{1{,}06^5 - 1}{1{,}06^5(1{,}06-1)}$

a = **50.000 €** $\quad P = \mathbf{210.618,19}$ €

P = ? $\quad I_1 = P_{k-1} \times \frac{j}{100} = 210.618{,}19 \times 0{,}06 = \mathbf{12.637,09}$ €

$R_1 = a - I_1 = 50.000 - 12.637{,}09 = \mathbf{37.362,91}$ €

$P_1 = P_{k-1} - R_1 = 210.618{,}19 - 37.362{,}91 = \mathbf{173.255,28}$ €

Time period	*Annuity*	*Interest*	*Payment quota*	*Outstanding principle*
K	**ak**	**Ik**	**Rk**	**Pk**
0	-	-	-	210.618,19
1	50.000,00	12.637,09	37.362,91	173.255,28
2	50.000,00	10.395,32	39.604,68	133.650,60
3	50.000,00	8.019,04	41.980,96	91.669,63
4	50.000,00	5.500,18	44.499,82	47.169,81
5	50.000,00	2.830,19	47.169,81	0,00
Σ	*250.000,00*	*39.381,81*	*210.618,19*	

3) The loan of EUR 50,000.00 was approved for 2 years with 10 % of decursive annual interest and payment of equal annuities semi-annually. Create a repayment table if the interest calculation is compound and semi-annual. Use the conformal interest calculation.

P = 50.000 € $\quad m = \frac{n_1}{n_2} = \frac{1}{1/2} = 2$

n = 2 years $\quad j' = 100 \times \left[\left(1 + {}^{j}/_{100}\right)^{\frac{1}{m}} - 1\right] = 100 \times \left[\left(1 + {}^{10}/_{100}\right)^{\frac{1}{2}} - 1\right] =$

$j' = 4{,}88\ \%$

j = 10 % → i = 1,1 $\quad i' = 1{,}0488$

n1 = 1 year $\quad a = P \times \frac{i^n(i-1)}{i^n - 1} = 50.000 \times \frac{1{,}0488^4(1{,}0488-1)}{1{,}0488^4 - 1} = 14.061{,}31$ €

n2 = 1/2 year $\quad I_1 = P_{k-1} \times \frac{j}{100} = 50.000 \times \frac{4{,}88}{100} = 2.440$ €

$R_1 = a - I_1 = 14.061{,}31 - 2.440 = 11.621,\ 31$ €

$P_1 = P_{k-1} - R_1 = 50.000 - 11.621{,}31 = 38.378{,}69$ €

Time period	*Annuity*	*Interest*	*Payment quota*	*Outstanding principle*
K	**ak**	**Ik**	**Rk**	**Pk**
0	-	-	-	50.000,00

Ivan Prudky, ivan.prudky@gmail.com

1	14.061,31	2.440,00	11.621,31	38.378,69
2	14.061,31	1.872,88	12.188,43	26.190,27
3	14.061,31	1.278,08	12.783,22	13.407,04
4	14.061,31	654,26	13.407,04	0,00
Σ	*56.245,23*	*6.245,23*	*50.000,00*	

Loans with variable annuity (periodic payment)

- *It is used when a loan user estimates that this repayment model will suit him – expecting the effects of future business*

a) *Equal payment quotas*
b) *Variable payment quotas*

Loan – variable annuity – equal payment quota

- *Loan P must be equal to the current value of n postumerando annuities:*
  - $P = a \times \frac{i^n - 1}{i^n(i-1)}$ → $a = P \times \frac{i^n(i-1)}{i^n - 1}$ → $a_k = I_k + R$
- *Interests are obtained from the rest of the debt from the previous period:*

$$I_k = P_{k-1} \times \frac{j}{100}$$

- *Payment quota Rk is a difference between annuity and interest:*

$$R = P_0 \,/\, n$$

- *The outstanding principal is the difference of previous debt and current payment quota:*

$$P_k = P_{k-1} - R_k$$

1) The company has been granted a loan of EUR 150,000 for a period of 5 years with a compound, annual and decursive interest calculation with the interest of 6%. The loan will be repaid with equal payment quotas at the end of the year. Create a repayment table.

P = 150.000 kn $\qquad R = \frac{P_0}{n} = \frac{150.000}{5} = 30.000\ €$

n = 5 years $\qquad I_1 = P_{k-1} \times \frac{j}{100} = 150.000 \times \frac{6}{100} = 9.000\ €$

j = 6 % → i = 1,06 $\qquad a_1 = I_1 + R = 9.000 + 30.000 = 39.000\ €$

R = ?

Time period	*Annuity*	*Interest*	*Payment quota*	*Outstanding principal*
K	**ak**	**Ik**	**Rk**	**Pk**
0	-	-	-	150.000,00
1	39.000,00	9.000,00	30.000,00	120.000,00
2	37.200,00	7.200,00	30.000,00	90.000,00
3	35.400,00	5.400,00	30.000,00	60.000,00

Ivan Prudky, ivan.prudky@gmail.com

4	33.600,00	3.600,00	30.000,00	30.000,00
5	31.800,00	1.800,00	30.000,00	0,00
Σ	*177.000,00*	*27.000,00*	*150.000,00*	

2) The bank approved a loan of EUR 300,000 for a period of 3 years with 5 % annual decursive interest and payment with equal payment quotas semi-annually. The calculation of interest is complex and semi-annual. Calculate the relative interest rate. Create a repayment table.

3) The bank approved the company a loan of EUR 175,000 for a period of 5 years with 3 % semi-annual decursive interest rates and payment with equal repayment quotas at the end of each year. The calculation of interest is complex and annual. Calculate the conformal interest. Create a repayment table.

P = 175.000 € $m = \frac{n_1}{n_2} = \frac{1/2}{1} = 1/2$

n = 5 years $j' = 100 \times \left[\left(1 + {}^{j}/_{100}\right)^{\frac{1}{m}} - 1\right] = 100 \times \left[\left(1 + {}^{3}/_{100}\right)^{\frac{1}{1/2}} - 1\right]$

j = 3 % $j' = 6{,}09\ \%$ $i' = 1{,}0609$

n1 = ½ year $R = \frac{P_0}{n} = \frac{175.000}{5} = 35.000\ €$

n2 = 1 year $I_1 = P_{k-1} \times \frac{j}{100} = 175.000 \times \frac{6{,}09}{100} = 10.657{,}50\ €$

$a_1 = I_1 + R = 10.657{,}50 + 35.000 = 45.657{,}50\ €$

Time period	*Annuity*	*Interest*	*Payment quota*	*Outstanding principal*
K	**ak**	**Ik**	**Rk**	**Pk**
0	-	-	-	175.000,00
1	45.657,50	10.657,50	35.000,00	140.000,00
2	43.526,00	8.526,00	35.000,00	105.000,00
3	41.394,50	6.394,50	35.000,00	70.000,00
4	39.263,00	4.263,00	35.000,00	35.000,00
5	37.131,50	2.131,50	35.000,00	0,00
Σ	*206.972,50*	*31.972,50*	*175.000,00*	

Loans with incomplete annuity

- *The incomplete annuity is calculated:*
  - *the last payment quota should be equal to the rest of the debt (outstanding principle) from the previous period,*
  - *the last payment quota summed up with last interest equals the incomplete annuity.*

k	***a***	***Ik***	***Rk***	***Pk***
0	–	–	–	*150.000,00*

1	45.000,00	18.000,00	27.000,00	123.000,00
2	45.000,00	14.760,00	30.240,00	92.760,00
3	45.000,00	11.131,20	33.868,80	58.891,20
4	45.000,00	7.066,94	37.933,06	**20.958,14**
5	**23.247,12**	**2.514,98**	**20.958,14**	
	203.473,12	53.473,12	150.000,00	

Loan conversion

- *in the course of the repayment of the loan, a situation in which a lender or borrower is required to change in one or more elements from the loan contract that they signed*
- *any such change is called a loan conversion*
- *a new amount of annuity is determined, which is calculated on the basis of the rest of the debt in the period when one or more changes occurred*

Loan conversion

- *Loan P must be equal to the current value of n postumerando annuities:*

$$P = a \times \frac{i^n - 1}{i^n(i-1)} \quad \rightarrow \quad a = P \times \frac{i^n(i-1)}{i^n - 1}$$

- *Interests are obtained from the rest of the debt from the previous period:*

$$I_k = P_{k-1} \times \frac{j}{100}$$

- *Payment quota Rk is a difference between annuity and interest:*

$$R_k = a - I_k \quad \rightarrow \quad R_k = R_1 \times i^{k-1}$$

- *The outstanding principal is the difference of previous debt and current payment quota:*

$$P_k = P_{k-1} - R_k$$

1) The company was approved for a loan of EUR 400,000 for 3 years with 12 % annual interest and equal annuities at the end of the year. After the payment of the second annuity, the repayment time is extended by an additional year. Construct the repayment table. The calculation of interest is annual, compound, and decursive.

2) The company was approved for a loan of EUR 150,000 for 2 years, with 7 % of the annual interest and semi-annual payment of equal annuities. After the payment of the second annuity, the repayment time extends by a year. Construct the repayment table. The calculation of interest is semi-annual, compound, and decursive. Use a relative interest rate.

P = 150.000 € $\quad a1 = P \times \frac{i^n(i-1)}{i^n-1} = 150.000 \times \frac{1{,}035^4(1{,}035-1)}{1{,}035^4-1} = \mathbf{40.837{,}67}$ €

n = 2 years $\quad I_1 = P_{k-1} \times \frac{j}{100} = 150.000 \times \frac{3{,}5}{100} = 5.250{,}00$ €

j = 7 % $\quad R_1 = a - I_1 = 40.837{,}67 - 5.250{,}00 = 35.587{,}67$ €

m = 1/(1/2) $\quad P_1 = P_{k-1} - R_1 = 150.0000 - 35.587{,}67 = 114.412{,}33$ €

jr = 3,5 % $a2 = P \times \frac{i^n(i-1)}{i^n-1} = 77.579{,}09 \times \frac{1{,}035^4(1{,}035-1)}{1{,}035^4-1} = \mathbf{21.121{,}00}$ €

Time period	*Annuity*	*Interest*	*Payment quota*	*Outstanding principal*
K	**ak**	**Ik**	**Rk**	**Pk**
0	-	-	-	150.000,00
1	40.837,67	5.250,00	35.587,67	114.412,33
2	40.837,67	4.004,43	36.833,24	77.579,09
3	21.121,00	2.715,27	18.405,73	59.173,36
4	21.121,00	2.071,07	19.049,93	40.123,43
5	21.121,00	1.404,32	19.716,68	20.406,76
6	21.121,00	714,24	20.406,76	0,00
Σ	*166.159,32*	*16.159,32*	*150.000,00*	

3) The company was approved for a loan of EUR 62,700 for 4 years, with 3% semi-annual interest and equal annuities at the end of each year. After the payment of the second annuity, the repayment time extends by a year. Construct the repayment table. The calculation of interest is annual, compound, and decursive. Calculate using the conformal interest rate.

P = 62.700 € $j' = 100 \times \left[\left(1 + {}^{j}/_{100}\right)^{\frac{1}{m}} - 1\right] = 100 \times \left[\left(1 + {}^{3}/_{100}\right)^{\frac{1}{1/2}} - 1\right] =$

$j' = 6{,}09\ \%$

n = 5 years $a1 = P \times \frac{i^n(i-1)}{i^n-1} = 62.700 \times \frac{1{,}0609^4(1{,}0609-1)}{1{,}0609^4-1} = \mathbf{18.131,\ 99}$ €

j = 3 % $I_1 = P_{k-1} \times \frac{j}{100} = 62.700 \times \frac{6{,}09}{100} = 3.818{,}43$ €

m = (1/2)/1 = 1/2 $R_1 = a - I_1 = 18.131{,}99 - 3.818{,}43 = 14.313{,}56$ €

$P_1 = P_{k-1} - R_1 = 62.700 - 14.313{,}56 = 48.386{,}44$ €

$a2 = P \times \frac{i^n(i-1)}{i^n-1} = 33.201{,}18 \times \frac{1{,}0609^3(1{,}0609-1)}{1{,}0609^3-1} = \mathbf{12.441{,}58}$ €

Time period	*Annuity*	*Interest*	*Payment quota*	*Outstanding principal*
K	**ak**	**Ik**	**Rk**	**Pk**
0	-	-	-	62.700,00
1	18.131,99	3.818,43	14.313,56	48.386,44
2	18.131,99	2.946,73	15.185,26	33.201,18
3	12.441,58	2.021,95	10.419,62	22.781,56
4	12.441,58	1.387,40	11.054,18	11.727,38
5	12.441,58	714,20	11.727,38	0,00
Σ	*73.588,71*	*10.888,71*	*62.700,00*	